Bernoulli-IMS One World Symposium 2020

An Eyring–Kramers law for periodically forced bistable systems

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INSTITUT ORLÉANS · TOURS Mathématiques & Physique Théorique DENIS POISSON

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partly based on joint work with Barbara Gentz (Bielefeld)

PROJET FINANCÉ PAR L'ANR - PROJECT FUNDED BY THE ANR

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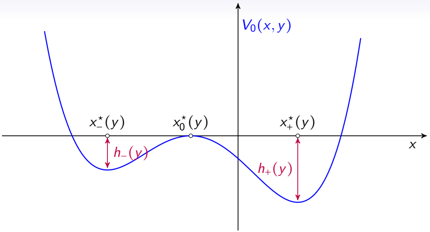
project PERISTOCH

The problem



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- $x \mapsto V_0(x,y)$ double-well potential, $V_0(x,y+1) = V_0(x,y)$
- $0 \leqslant \varepsilon, \sigma \ll 1$
- W_t^x, W_t^y independent standard Wiener processes

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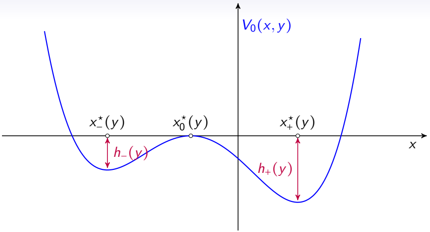

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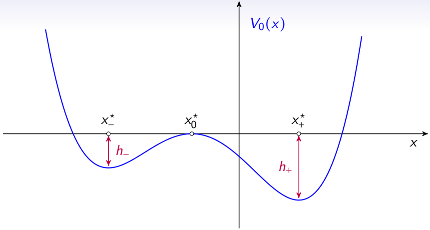
Question: law of $\tau_+ = \inf\{t > 0 \colon x_t = x^*_+(y_t) | (x_0 = x^*_-(y_0), y_0)\}$

Static case

$$\mathrm{d}x_t = -V_0'(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

$$\omega_\pm = \sqrt{V_0''(x_\pm^*)} \quad \omega_0 = \sqrt{-V_0''(x_0^*)}$$



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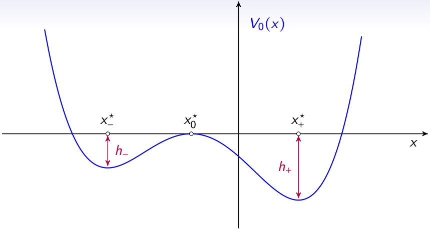

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$$\mathbb{E}^x[\tau_+] = \frac{2}{\sigma^2} \int_x^{x_+^*} \int_{-\infty}^{x_2} e^{2[V_0(x_2)-V_0(x_1)]/\sigma^2}\,dx_1\,dx_2$$

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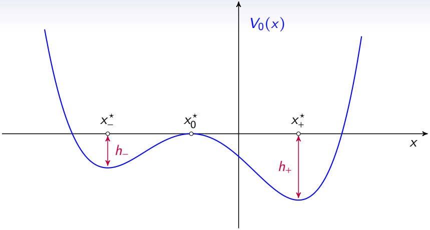

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$\Rightarrow$ Eyring–Kramers law: $\mathbb{E}^{x_-^*}[\tau_+] = \dfrac{2\pi}{\omega_0\omega_-}\,\mathrm{e}^{2h_-/\sigma^2}\,[1+\mathcal{O}(\sigma^2)]$

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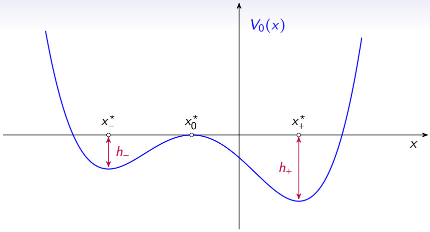

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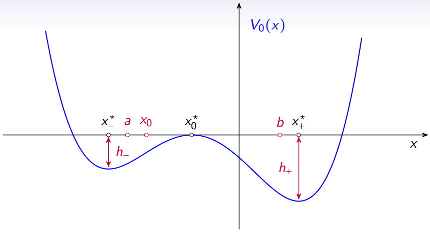

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▷ [Cérou, Guyader, Lelièvre, Malrieu 13]: Reactive path $x_-^* < a < x_0 < x_0^* < b < x_+^*$

$$\lim_{\sigma\to 0} \mathrm{Law}\big(\omega_0\tau_b - 2\log(\sigma^{-1}) \,\big|\, \tau_b < \tau_a\big) = \mathrm{Law}\big(\underbrace{\mathcal{G}}_{\text{Gumbel}} + \underbrace{T(x_0,b)}_{\text{deterministic}}\big)$$

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  - [Bouchet & Reygner 2016]: Formal computations → Eyring–Kramers law in bistable situations
  - [Landim, Mariani & Seo 2019]: Non-reversible potential theory
    Confirms result by [B & R 2016] for some systems with known π
  - [Le Peutrec & Michel 2019]: Semiclassical analysis for systems with known π

Back to the problem

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τ_0 hitting time of (periodic orbit tracking) saddle $x_0^*(y)$

Theorem: [B & Gentz, SIAM J Math Analysis 2014]

$$\lim_{\sigma\to 0} \mathrm{Law}\Big(\theta(y_{\tau_0}) - \log(\sigma^{-1}) - \frac{\lambda_+}{\varepsilon} Y^\sigma\Big) = \mathrm{Law}\Big(\frac{\mathscr{G}}{2} - \frac{\log 2}{2}\Big)$$

- $\theta(y)$: explicit parametrisation of periodic orbit tracking $x_0^*(y)$
- λ_+: Lyapunov exponent of periodic orbit
- $Y^\sigma \in \mathbb{N}$: period during which transition occurs

$$\lim_{n\to\infty} \mathbb{P}\{Y^\sigma = n+1 | Y^\sigma > n\} = p(\sigma)$$

$p(\sigma) \simeq \mathrm{e}^{-I/\sigma^2}$ where I quasipotential

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$$\begin{aligned} dx_t &= -\partial_x V_0(x_t, y_t)\, dt + \sigma\, dW_t^x \\ dy_t &= \varepsilon\, dt + \sigma\sqrt{\varepsilon}\varrho\, dW_t^y \end{aligned}$$

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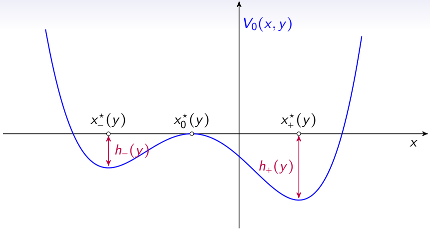
$\mathbb{E}[\tau_0], \mathbb{E}[\tau_+] \sim p(\sigma)^{-1}$ but how about sharp asymptotics?

New result

$$\omega_\pm(y) = \sqrt{\partial_{xx} V_0(x^*_\pm(y), y)}$$

$$\omega_0(y) = \sqrt{-\partial_{xx}(x^*_0(y), y)}$$

$$r_\pm(y) = \frac{\omega_\pm(y)\omega_0(y)}{2\pi}\, e^{-2h_\pm(y)/\sigma^2}$$



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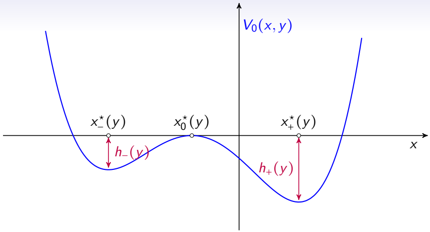

- ▷ Leading eigenvalue of $-\mathscr{L}_x = -\frac{\sigma^2}{2}\partial_{xx} + \partial_x V_0 \partial_x$:

  $\lambda_1(y) = [r_+(y) + r_-(y)][1 + \mathcal{O}(\sigma^2)]$

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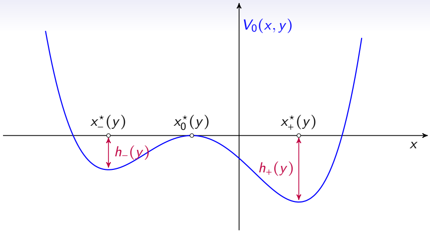

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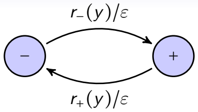
$$\lambda_1(y) = [r_+(y) + r_-(y)][1 + \mathcal{O}(\sigma^2)] \qquad \langle\lambda_1\rangle = \int_0^1 \lambda_1(y)\,\mathrm{d}y$$

Theorem: [B 2020, arXiv:2007.08443]

$$\mathbb{E}^{(x^*_-(y_0),y_0)}[\tau_+] = \frac{2\pi[1 + R(\varepsilon,\sigma)]}{\displaystyle\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y}$$

where $R(\varepsilon,\sigma)$ complicated but small if $\langle\lambda_1\rangle \ll \varepsilon \ll \langle\lambda_1\rangle^{1/4}$

Intuition: two-state jump process



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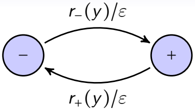

$$R_-(y_1, y_0) = \int_{y_0}^{y_1} r_-(y)\,\mathrm{d}y$$

$$\mathbb{E}^{-,y_0}\big[\tau_+\big] = \frac{\displaystyle\int_0^1 \mathrm{e}^{-R_-(y_0+y,y)/\varepsilon}\,\mathrm{d}y}{1 - \mathrm{e}^{-R_-(1,0)/\varepsilon}} \simeq \begin{cases} \dfrac{\varepsilon}{R_-(1,0)} & \text{if } \varepsilon \gg \max\limits_{y\in[0,1]} r_-(y) \\ \dfrac{\varepsilon}{r_-(y_0)} & \text{if } \varepsilon \ll \min\limits_{y\in[0,1]} r_-(y) \end{cases}$$

In between: Stochastic resonance

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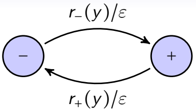

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In between: Stochastic resonance

Principle of proof:

- Non-reversible potential theory [Landim, Mariani & Seo 2019]:

$$\int_{\partial\mathcal{A}} \mathbb{E}^x\bigl[\tau_{\mathcal{B}}\bigr]\,\mathrm{d}\nu_{\mathcal{A}\mathcal{B}} = \frac{1}{\operatorname{cap}(\mathcal{A},\mathcal{B})} \int_{\mathcal{B}^c} h^*_{\mathcal{A}\mathcal{B}}\,\mathrm{d}\pi$$

- Capacity $\operatorname{cap}(\mathcal{A},\mathcal{B})$ obeys variational principles
- Main difficulty: control invariant measure $\pi(x,y)$
  Use decomposition on eigenbasis of $\mathscr{L}_x^\dagger$

References

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- N. B., *Noise-induced phase slips, log-periodic oscillations, and the Gumbel distribution*, Markov Process. Related Fields, **22**(3):467–505, 2016
- C. Landim, M. Mariani, & I. Seo, *Dirichlet's and Thomson's principles for non-selfadjoint elliptic operators with application to non-reversible metastable diffusion processes*, Arch. Ration. Mech. Anal., **231**(2):887–938, 2019
- N. B., *An Eyring–Kramers law for slowly oscillating bistable diffusions*, Preprint, July 2020, `arXiv:2007.08443`

Thanks for watching!

Slides available at https://www.idpoisson.fr/berglund/BIOWS20.pdf