A UNIVERSAL MODEL FOR THE BIFURCATIONS OF ASYMPTOTIC VALUES

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Abstract. We study the notion of tangent-like maps, which is a transcendental analogue of polynomial-like maps. We introduce a model family analogous to quadratic polynomials, with only one free asymptotic value, and define the "Tandelbrot set" as the analogue of the Mandelbrot set. We prove a Straightening Theorem for tangent-like maps, with uniqueness of the model map in the case where the filled-in Julia set is connected, and a parameter version of the Straightening Theorem for suitable holomorphic families of tangent-like maps. As a consequence, we prove the existence of topological copies of the Tandelbrot set in bifurcation loci of numerous families of meromorphic maps.

Contents

1. Introduction

The quadratic family $f_c(z) := z^2 + c$, $c \in \mathbb{C}$ is the simplest non-trivial holomorphic family of rational maps in one complex variable. These maps have only one critical point, located at $z = 0$, which is the only point at which f_c fails to be a local homeomorphism. The orbit of $z = 0$ governs the global dynamics of f_c. The Mandelbrot set $\mathcal{M}$, defined as the set of parameters $c \in \mathbb{C}$ for which the orbit of $z = 0$ does not escape to infinity, has been the subject of extensive research over the last decades and still is the object of one of the main open questions in the field: the MLC conjecture, which states that $\mathcal{M}$ is locally connected. It is known that the MLC conjecture would imply a positive answer to several other important conjectures, such as the Fatou conjecture on the genericity of hyperbolicity in the quadratic family.

A remarkable feature of the Mandelbrot set is its *universality*. In the early 80's, many numerical experiments showed the existence of topological copies of quadratic Julia sets in the dynamical plane of polynomial or rational maps, or copies of the Mandelbrot set in the parameter spaces of most holomorphic families of rational maps. Both of these phenomena are explained by the seminal work of Douady and Hubbard on polynomial-like mappings.

Introduced in [DH85], polynomial-like mappings provide a powerful conceptual framework that allows one to transfer results from polynomial dynamics to a much broader class of holomorphic dynamical systems. A polynomial-like map is a triple (f, U, V) where U and V are topological disks, $\overline{U} \subset V$, and $f : U \to V$ is a proper holomorphic map of degree $d \geq 2$. The fundamental result of this theory is the so-called Straightening Theorem ([DH85, Theorem 1]), which asserts that every polynomial-like map is (quasiconformally) conjugate to a genuine polynomial P in a neighborhood of their respective filled Julia sets, see Definition 2.8 for a more precise statement. On the other hand, given a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the flexibility of this definition allows in many situations to construct restrictions $f^n_{|U} : U \to V$ of some iterate of f which are quadratic-like, or more generally polynomial-like. The Straightening Theorem then explains why copies of polynomial Julia sets are encountered in the phase space of many rational or transcendental maps, near a critical point. In the particular case of a quadratic polynomial with connected Julia set, if such a restriction exists, then $f_c^n : U \to V$ is quasiconformally conjugated to a unique $f_{c'}$, $c' \in \mathcal{M}$: this gives rise to a *renormalization operator* $\mathcal{R} : c \mapsto c'$. The study of the fine properties of this renormalization operator are closely related to the MLC conjecture, by the famous work of Yoccoz ([Hub92]). We refer the reader the recent survey [Dud25] for an overview of this topic, and the related works of Avila, Kahn, Lyubich, McMullen, Shen and many others.

There is also a parameter version of the Straightening Theorem, which applies only to families of unicritical polynomial-like maps (in particular, for families of quadratic-like maps). It allowed McMullen ([McM00]) to prove the universality property of the Mandelbrot set mentionned above: in any holomorphic family of rational maps with non-empty bifurcation locus and simple critical points, there are quasiconformal copies of the Mandelbrot set, or of its higher degree analogue in the case of critical points with persistent higher multiplicity.

In a different setting, there has been remarkable progress during the recent years in the understanding of the dynamics of maps with essential singularities, an area known as transcendental dynamics. In the presence of critical points, transcendental functions behave locally like polynomials and we see polynomial Julia sets in their dynamical plane, as explained above. But a different type of singularity of the inverse map may exist for transcendental functions, namely *asymptotic values*. Heuristically speaking, asymptotic values v are points that have some of their preimages "at the essential singularity". More formally, given a holomorphic map $f : U \to V$ between Riemann surfaces, a point $v \in V$ is an asymptotic value for f if there exists a continuous curve $\gamma : [0, +\infty) \to U$ such that $\gamma(t)$ leaves every compact of U as $t \to +\infty$, but $\lim_{t\to+\infty} f \circ \gamma(t) = v$. The presence of critical values (that is, images of critical points) or asymptotic values is the only obstruction to the existence of well-defined local inverse branches of the map f.

A natural question one may ask is whether there exists a class of holomorphic maps analoguous to quadratic-like maps, but for asymptotic values instead of simple critical points. This leads to the notion of a so-called *tangent-like map*, of which we give here a first, informal definition: a holomorphic map $f : U \to V$ is tangent-like if there exists $v \in V$ such that $f : U \to V \setminus \{v\}$ is a universal cover, where U, V are Jordan domains and $U \Subset V$. Then v is an asymptotic value, and there will be an essential singularity $u \in \partial U$. See Definition 3.1. This notion was introduced by Galazka and Kotus ([GK08]).

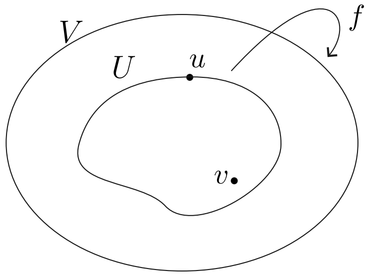

FIGURE 1. The definition of a tangent-like map. The point $u \in \partial U$ is an essential singularity. The asymptotic value v can be located anywhere in V.

Similarly to polynomial-like maps, one can define the *filled Julia set* $K(f)$ of a tangent-like map f as the points that either never leave U under iteration, or eventually land on the essential singularity u. The Julia set $J(f)$ is defined as its boundary of $K(f)$. The connectedness of $K(f)$ or $J(f)$ can be proved to be equivalent to the property $v \in K(f)$ (Proposition 3.7). Likewise, the concept of quasiconformal conjugacy and hybrid equivalence can be appropriately defined in this setting (see Section 3 for details),

Note that unless $u = v$, tangent-like maps have points which are mapped to an essential singularity. Therefore, the class of entire maps is somehow too restrictive to admit many different tangent-like restrictions; instead, the natural setting is the class of transcendental meromorphic maps, and their iterates.

We now introduce a model family, which will be our analogue of the quadratic family:

$$T_\alpha(z) := \frac{e^{(\alpha-1)z/8} - 1}{\alpha e^{(\alpha-1)z/8} - 1}, \qquad \alpha \in \mathbb{C} \setminus \{1\}. \tag{1.1}$$

For $\alpha \in \mathbb{C} \setminus \{1\}$, the map T_α has exactly two asymptotic values (which are 1 and $\frac{1}{\alpha}$), no critical points and it also fixes 0 with multiplier equal to $\frac{1}{8}$. These properties characterize the family, up to affine conjugation (see Lemma 4.2). Putting aside the choice of multiplier $\frac{1}{8}$, which is arbitrary, this is in a way the simplest possible family of meromorphic maps with a free asymptotic value. Maps in this slice were previously studied in [GK08, Ga11, CJK22] and [CJK23].

This family exhibits a symmetry with respect to the involution $\alpha \mapsto \frac{1}{\alpha}$, in the sense that T_α is linearly conjugated to $T_{1/\alpha}$, and the conjugacy switches the two asymptotic values. We shall therefore mostly be concerned with values of α in $\overline{\mathbb{D}} \setminus \{1\}$, and work with the free asymptotic value $\frac{1}{\alpha}$. Indeed, for $\alpha \in \overline{\mathbb{D}} \setminus \{1\}$, one can show (Lemma 4.1) that the asymptotic value 1 is always in the basin of 0, while $1/\alpha$ is free and may or may not be captured by this basin. The basin of 0, which we denote by Ω_α, will play the same role as the basin of infinity for polynomials.

Our main motivation for introducing the family $(T_\alpha)_{\alpha \in \mathbb{C} \setminus \{1\}}$ is the following straightening theorem. The first part of this result was already anticipated by Galazka and Kotus in [GK08], though the argument given there appears to be incomplete.

Theorem A (Straightening Theorem)**.** *Let (f, U, V, u, v) be a tangent-like map. Then there exists $\alpha \in \mathbb{C} \setminus \{1\}$ such that f is hybrid equivalent to T_α, with the hybrid conjugacy mapping v to $\frac{1}{\alpha}$. Moreover, if $K(f)$ is connected, then α is unique.*

See Figure 2.

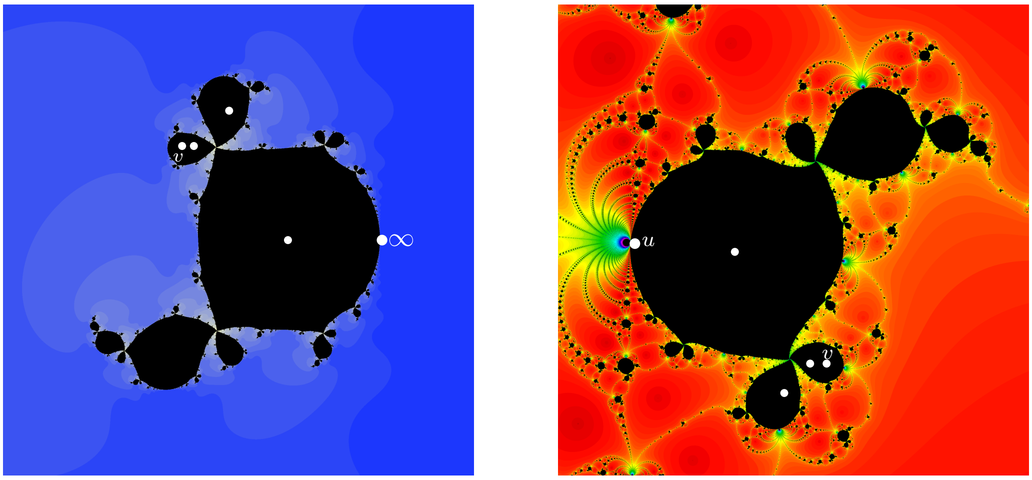

FIGURE 2. Left: The filled Julia set $K(T_\alpha)$ of T_α for $\alpha \simeq -0.021 + i0.009$, showing the basin of attraction of an attracting 3-periodic orbit. The asymptotic value is $v = 1/\alpha$. Right: A quasiconformal copy of $K(T_\alpha)$ in the dynamical plane of a Newton's method N_a, applied to the family of entire functions $f_a(z) = z + ae^z$ for $a \simeq -1.1627 + i0.1143$. There is a 3-periodic orbit under N_a^2. The point u is a pole of N_a which becomes an essential singularity under the second iterate. The assymptotic value is $v = 0$.

We define the *filled Julia set* of T_α as $K(T_\alpha) := \hat{\mathbb{C}} \setminus \Omega_\alpha$. The Julia set is the usual set of non-normality. As we will see in Section 4, the maps T_α all admit tangent-like restrictions, and the definitions of filled-Julia set and Julia sets agree with those given above for tangent-like maps.

By analogy with the Mandelbrot set, we define the *Tandelbrot set* $\mathcal{T}$:

$$\mathcal{T} := \{\alpha \in \mathbb{C} \setminus \{1\} : \; T_\alpha^n \left(\frac{1}{\alpha}\right) \not\to 0 \text{ as } n \to \infty\} \tag{1.2}$$

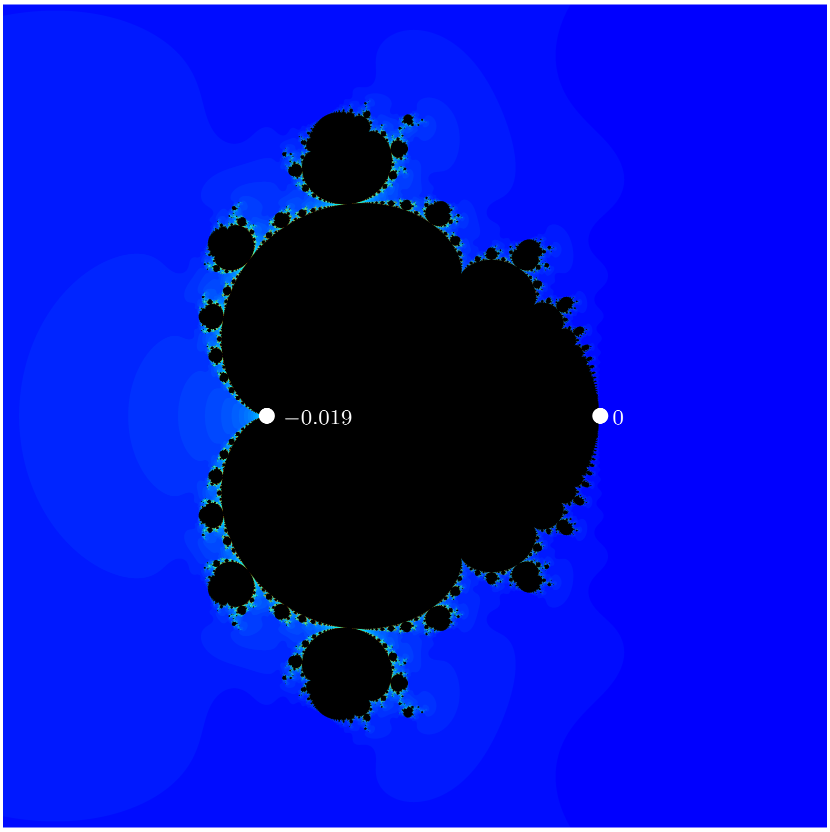

FIGURE 3. In black, the Tandelbrot set.

Theorem B (Properties of $\mathcal{T}$)**.** *The Tandelbrot set $\mathcal{T}$ is a full and connected compact set, contained in $\mathbb{D}$. Its boundary is the bifurcation locus of the family $\{T_\alpha\}_{\alpha \in \mathbb{D}}$. Moreover, for all $\alpha \in \mathbb{D}$, $\alpha \in \mathcal{T}$ if and only if $J(T_\alpha)$ is connected.*

The assertion that $\mathcal{T}$ is connected and full was already proved in [CJK23]. We will give here a different, shorter proof (see Theorem 4.19).

Recall that a holomorphic family of meromorphic maps $\{f_\lambda\}_{\lambda \in M}$ is called *natural* if there exists quasiconformal homeomorphisms $\varphi_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $\psi_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$, for some $\lambda_0 \in \Lambda$, see Section 2.3. The activity locus $\mathcal{A}(v_{\lambda_0})$ of a marked asymptotic value $\lambda \mapsto v_\lambda := \varphi_\lambda(v_{\lambda_0})$ is by definition the non-normality locus of $\{\lambda \mapsto f_\lambda^n(v_\lambda), n \in \mathbb{N}\}$. We also refer to Section 2.3 the link between activity loci and J-stability, and to Section 2.4 the definition of a *tract over an asymptotic value*. For every $\lambda ** \in M$ and any one-dimensional topological disc $D \subset M$ containing λ_*, the natural family can be reparametrized in D, so that it is defined for $\lambda \in \mathbb{D}$, with basepoint $f_{\lambda_*} = f_0$. Hence the following is actually a local theorem, which states that copies of the Tandelbrot set can be found arbitrarily close to any parameter λ_* in the activity locus of an asymptotic value (satisfying certain conditions).

Theorem C (Universality of the Tandelbrot set). *Let $\{f_\lambda\}_{\lambda\in\mathbb{D}}$ be a natural family of meromorphic maps, and assume that the following conditions hold:*

(1) f_0 has an active isolated asymptotic value v_0, whose forward orbit does not persistently contain a critical point;
(2) f_0 has a tract above v_0 which is a quasidisk;
(3) and f_0 has at least one simple pole which is not a singular value.

Then there exists a topological embedding $\chi : \partial\mathcal{T} \to \mathcal{A}(v_0)$.

As a consequence of Theorem C, and the results proved by McMullen in [McM00], copies of either Mandelbrot sets or Tandelbrot sets are dense in bifurcation loci of natural families of finite type meromorphic maps.

Definition 1.1 (Multibrot sets). The degree $d \geq 2$ Mandelbrot set $\mathcal{M}_d$ is defined by

$$\mathcal{M}_d := \{\lambda \in \mathbb{C} : J(z \mapsto z^d + \lambda) \text{ is connected}\}.$$

Corollary 1.2. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a natural family of finite type meromorphic maps. Assume that the following conditions hold:*

(1) No critical point is persistently a Picard exceptional value.
(2) All tracts of the maps f_λ above quasidisks are quasidisks.
(3) For all $\lambda \in \Lambda$, there exists a simple pole p_λ which is not a singular value.

Then, near any $\lambda_1 \in$ Bif, there exists a topological embedding of $\partial\mathcal{M}_d$ (for some $d \geq 2$) or $\partial\mathcal{T}$ contained in Bif.

Another immediate consequence of Theorem C is the following self-similarity property of the Tandelbrot set, similarly to the Mandelbrot set. See Figure 5.

Corollary 1.3 (Self-similarity of $\mathcal{T}$). *For any open set W intersecting $\partial\mathcal{T}$, there is a topological embedding $\chi : \partial\mathcal{T} \to W \cap \partial\mathcal{T}$.*

One of the two main ingredients in the proof of Theorem C is a way to construct a one-parameter family of tangent-like maps from suitable restrictions of some iterates of the maps f_λ. In [McM00], the mechanism leading to the construction of polynomial-like restrictions is a careful analysis of perturbations of maps with a Misiurewicz relation, i.e. maps which have a critical point whose forward orbit eventually lands on a repelling cycle. In our case, Misiurewicz relations are replaced with so-called *virtual cycles*, i.e. asymptotic values whose forward orbit eventually lands on an essential singularity. See Figure 4. Virtual cycles are linked to bifurcations of families of meromorphic maps, see Section 2 for a short review of the results from [ABF21], [ABF24]. Assumption (2) in Theorem C is necessary to ensure the existence of these tangent-like restrictions. Additionnally, the existence of non-constant families of tangent-like restrictions imply the existence of poles, which is why we need assumption (3). This explains why topological copies of $\partial\mathcal{T}$ are not observed in pictures of parameter spaces of entire maps (such as the exponential family $f_\lambda(z) := \lambda e^z$ for instance).

The second main ingredient in the proof of Theorem C is the following parameter version of the Straightening Theorem, which is of independent interest. Informally, a tangent-like family is a holomorphic family of tangent-like maps $\{f_\lambda : U_\lambda \to V_\lambda\}_{\lambda\in\Lambda}$. As for families of quadratic-like maps, we say that such a family is *proper* if Λ is a Jordan domain, if $\{f_\lambda\}_{\lambda\in\Lambda}$ is a restriction of another tangent-like family defined over a slightly larger parameter space $\widetilde{\Lambda}$ and

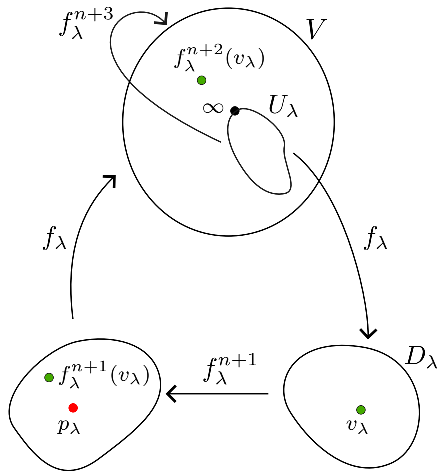

FIGURE 4. Sketch of the tangent-like renormalization scheme. Suppose λ_0 is a virtual cycle parameter for which $f_{\lambda_0}^n(v_{\lambda_0}) = \infty$. Then, for nearby parameters λ the domain U_λ of the tangent-like restriction is a tract above a disk D_λ containing the asymptotic value v_λ. The iterate f_λ^{n+1} maps D_λ to another disk containing the pole p_λ, which is then mapped to a fixed disk V centered at infinity. Hence, the map $f_\lambda^{n+3} : U_\lambda \to V \setminus \{f_\lambda^{n+2}(v_\lambda)\}$ is tangent-like.

phase spaces $\widetilde{U}_\lambda$, and if the asymptotic value v_λ winds once around the essential singularity as λ turns around $\partial\Lambda$. We say that it is *equipped* if there is a holomorphic motion of the fundamental annulus $\overline{V_\lambda} \setminus U_\lambda$ commuting with f_λ. See Definitions 6.1, 6.2 and 6.3.

Theorem D (Parameter version of the Straightening Theorem)**.** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a proper equipped tangent-like family. Let*

$$\mathcal{T}_0 = \{\lambda \in \Lambda : v_\lambda \in K(f_\lambda)\}.$$

Then, there exists a continuous map $\chi : \Lambda \to \mathbb{C} \setminus \{1\}$ such that

(1) For all $\lambda \in \Lambda$, f_λ is hybrid equivalent to $T_{\chi(\lambda)}$.
(2) $\chi : \mathcal{T}_0 \to \mathcal{T}$ is a homeomorphism.
(3) χ is holomorphic on $\mathring{\mathcal{T}}_0$.

See Fig. 5.

Organization of the paper

We begin by recalling some background results on holomorphic motions and polynomial-like maps in Section 2, and some of the results from [ABF21]. In Section 3, we define tangent-like maps, hybrid equivalence and we prove some general properties of tangent-like maps. In Section 4, we introduce the model family $\{T_\alpha\}_{\alpha\in\mathbb{C}}$, and prove Theorem B. Section 5 is devoted to the proof of Theorem A. In Section 6, we define tangent-like families and we prove Theorem D. Finally, Section 7 is devoted to the proof of Theorem C and its Corollary.

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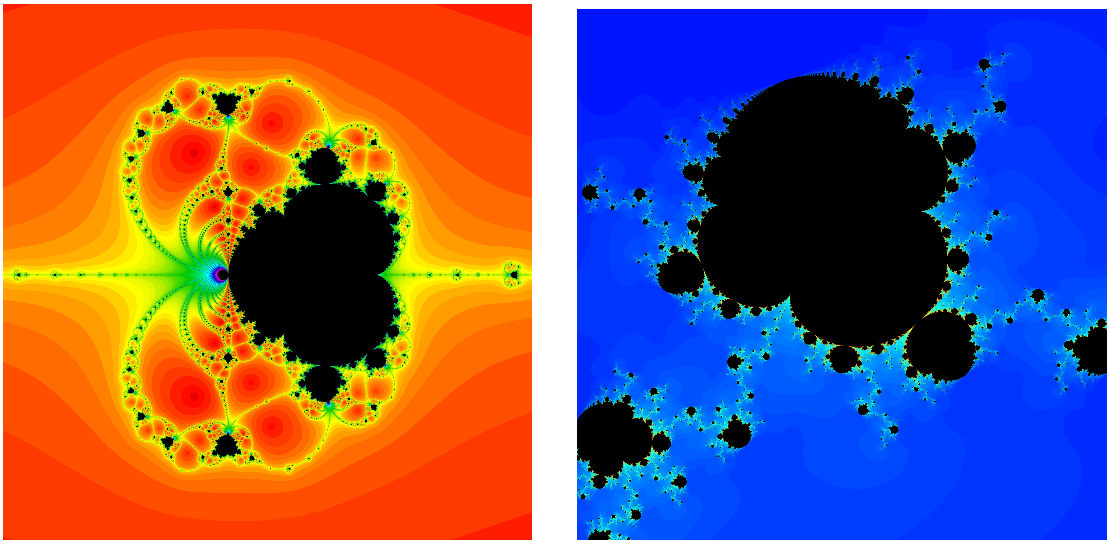

FIGURE 5. Left: A copy of $\mathcal{T}$ in the parameter space of the Newton's method of $f_a(z) = z + ae^z$, illustrating Theorem C. Right: A small copy of $\mathcal{T}$ inside $\mathcal{T}$, illustrating Theorem D.

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Notations

Troughout the paper, $\mathbb{C}$ denotes the complex plane, $\mathbb{D}$ the unit disk, $\hat{\mathbb{C}}$ the Riemann sphere, $\mathbb{H}$ the upper half plane, and S^1 the unit circle.

2. Preliminaries

2.1. **Holomorphic motions, quasiconformal maps.** We start by collecting some technical facts about holomorphic motions and quasiconformal homeomorphisms, that will be used throughougt the paper.

Definition 2.1 (Holomorphic motion)**.** A *holomorphic motion* of a set $X \subset \hat{\mathbb{C}}$ over a complex manifold U with basepoint $\lambda_0 \in U$ is a map $H : U \times X \to \hat{\mathbb{C}}$ given by $(\lambda, x) \mapsto H_\lambda(x)$ such that

(1) for each $x \in X$, $H_\lambda(x)$ is holomorphic in λ,
(2) for each $\lambda \in U$, $H_\lambda(x)$ is an injective function of $x \in X$, and,
(3) at λ_0, $H_{\lambda_0} \equiv \mathrm{Id}$.

We will make repeated use of the following classical extension theorems on holomorphic motions:

Proposition 2.2 (Mañé-Sad-Sullivan's λ-lemma, [MSS83])**.** *Let $H : U \times X \to \hat{\mathbb{C}}$ be a holomorphic motion of a set $X \subset \hat{\mathbb{C}}$ over U. Then H extends uniquely to a holomorphic motion $H : U \times \overline{X} \to \hat{\mathbb{C}}$ of the closure of X over U.*

The following much deeper result is due to Slodkowski.

Theorem 2.3 (Slodkowski's extension [Slo91])**.** *Let $H : U \times X \to \hat{\mathbb{C}}$ be a holomorphic motion of a set $X \subset \hat{\mathbb{C}}$ over a domain $U \subset \hat{\mathbb{C}}$. Then H extends to a holomorphic motion $H : U \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Moreover, for all $\lambda \in U$, $h_\lambda := H(\lambda, \cdot) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal homeomorphism.*

We shall use the properties of the boundary extension of maps defined in domains.

Lemma 2.4 (Boundary extension of conformal maps of $\mathbb{D}$ [Pom92, Sect. 5.1])**.** *Let $D \subset \mathbb{C}$ be a quasidisk, and let $\varphi : \mathbb{D} \to D$ be a Riemann map. Then φ extends continuously to S^1, and $\varphi : S^1 \to \partial D$ is quasisymmetric.*

It follows that if D_1, D_2 are quasidisks and $\varphi : D_1 \to D_2$ is conformal, then φ extends quasisimmetrically as a map from ∂D_1 to ∂D_2.

Lemma 2.5 (Interpolation in quasi-annuli [EFGPS26, Theorem 3.4])**.** *Let U, V, U', V' be Jordan domains, with $U \Subset V$ and $U' \Subset V'$. Let $\psi_2 : \partial U \to \partial U'$ and $\psi_1 : \partial V \to \partial V'$ be quasisymmetric homeomorphisms. Then there exists a quasiconformal homeomorphism $\psi : \overline{V} \setminus U \to \overline{V'} \setminus U'$ such that $\psi_{|\partial U} = \psi_2$ and $\psi_{|\partial V} = \psi_1$. Moreover, the quasiconformality constant of ψ depends only on the moduli of the annuli $\overline{V} \setminus U$ and $\overline{V'} \setminus U'$ and the quasisymetric constants of ψ_1 and ψ_2.*

2.2. **Polynomial-like maps.** Next, we recall some basic definitions about polynomial-like maps. General references are [DH85] and [Lyu24].

Definition 2.6 (Polynomial-like map)**.** A polynomial-like map is a proper holomorphic map $f : U \to V$, where U, V are Jordan domains and $U \Subset V$.

Up to restricting a little bit V and U, we may always assume without loss of generality that f extends continuously to $\overline{U}$.

In particular, a polynomial-like map f is a branched cover; it therefore has a topological degree $d \in \mathbb{N}$, and if $d \geq 2$, then it must have at least one critical point. We will say that a polynomial-like map is unicritical if it has exactly one critical point.

Definition 2.7 (Filled Julia set)**.** Let $f : U \to V$ be a polynomial-like map. The filled Julia set of f is $K(f) := \bigcap_{n \geq 0} f^{-n}(U)$. The Julia set of f is $J(f) := \partial K(f)$.

Definition 2.8 (Hybrid equivalence)**.** Let (f_1, U_1, V_1) and (f_2, U_2, V_2) be two polynomial-like. We say that they are hybrid equivalent if there exists a quasiconformal homeomorphism $\varphi : V_1 \to V_2$ such that for all $z \in U_1$,

$$\varphi \circ f_1(z) = f_2 \circ \varphi(z)$$

and moreover $\overline{\partial}\varphi = 0$ a.e. on $K(f_1)$.

The relation between polynomial-like maps and actual polynomials is settled in the celebrated Straightening Theorem.

Theorem 2.9 (Straightening Theorem for PL maps, [DH85])**.** *Let $f : U \to V$ be a polynomial-like map. There exists a polynomial P such that f is hybrid equivalent to P. Moreover, if $K(f)$ is connected, then P is unique up to affine conjugacy.*

This relation extends also to parameter spaces.

Definition 2.10 (Polynomial-like family, c.f. [Lyu24, p. 545])**.** Let $\Lambda \subset \mathbb{C}$ be a domain. A holomorphic family of unicritical polynomial-like maps is a family of polynomial-like maps $f_\lambda : U_\lambda \to V_\lambda$, $\lambda \in \Lambda$, such that:

(1) There exists $d \geq 2$ such that for all $\lambda \in \Lambda$, f_λ is unicritical of degree d
(2) $\mathcal{U} := \{(\lambda, z) : z \in U_\lambda\}$ and $\mathcal{V} := \{(\lambda, z) : z \in V_\lambda\}$ are domains in $\mathbb{C}^2$
(3) $\mathbf{f} : \mathcal{U} \to \hat{\mathbb{C}}$ defined by $\mathbf{f}(\lambda, z) = f_\lambda(z)$ is holomorphic.

Definition 2.11 (Equipped family)**.** A holomorphi holomorphic family of unicritical polynomial-like maps is *equipped* if there exists $\lambda_0 \in \Lambda$ and a holomorphic motion $h_\lambda : \overline{V_{\lambda_0}} \setminus U_{\lambda_0} \to \overline{V_\lambda} \setminus U_\lambda$ such that for all $\lambda \in \Lambda$, for all $z \in \partial U_{\lambda_0}$,

$$h_\lambda \circ f_\lambda(z) = f_\lambda \circ h_\lambda(z).$$

The holomorphic motion $(h_\lambda)_{\lambda \in \Lambda}$ is called *the equivariant holomorphic motion* of the fundamental annulus.

Definition 2.12 (Proper family)**.** A holomorphic family of unicritical polynomial-like maps is proper if:

(1) Λ is a Jordan domain, and $\{f_\lambda\}_{\lambda \in \Lambda}$ is the restriction of a slightly larger holomorphic family of unicritical polynomial-like maps $\{f_\lambda\}_{\lambda \in \widetilde{\Lambda}}$, where $\Lambda \Subset \widetilde{\Lambda}$.
(2) For all $\lambda \in \partial \Lambda$, $v_\lambda \in \partial V_\lambda$.
(3) The winding number of $f_\lambda(c_\lambda) - c_\lambda$ is 1, as λ turns once along $\partial \Lambda$ and where c_λ is the unique critical point of f_λ.

In [Lyu24], the third condition in the above definition is called being *unfolded.*

The following technical lemma will be useful in our proof of the connectivity of the Tandelbrot set $\mathcal{T}$. It is proved in [Lyu24] for quadratic-like families, but the proof is valid for families of unicritical polynomial-like maps.

Lemma 2.13 ([Lyu24, p. 549])**.** *Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a proper equipped holomorphic family of unicritical polynomial-like maps. Let $P_n := \{\lambda \in \Lambda : f_\lambda^n(c_\lambda) \in U_\lambda\}$. Then P_n is a Jordan domain.*

2.3. **Meromorphic maps, families and bifurcations.** We say that $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a *meromorphic* map if it is holomorphic and ∞ is an essential singularity.

In this section we collect properties of natural families meromorphic maps and results about their bifurcations. A standard reference for iteration of meromorphic maps is [Ber93] and stability results for natural family can be found in [ABF21, ABF24].

A point $v \in \hat{\mathbb{C}}$ is a critical value if it is the image of a critical point, i.e. if there exists $c \in \mathbb{C}$ such that $f'(c) = 0$ and $f(c) = v$. A point $v \in \hat{\mathbb{C}}$ is an asymptotic value if there exists a curve $\gamma : [0, +\infty) \to \mathbb{C}$ such that $\lim_{t \to +\infty} \gamma(t) = \infty$ and $\lim_{t \to +\infty} f \circ \gamma(t) = v$.

The singular value set of f, $S(f)$, is the closure of the set of critical or asymptotic values. It may also be caracterized as the smallest closed set $S \subset \hat{\mathbb{C}}$ such that $f : \mathbb{C} \setminus f^{-1}(S) \to \hat{\mathbb{C}} \setminus S$ is a covering map.

We say that f is of *finite type* if $S(f)$ is finite. In that case, $S(f)$ contains only critical or asymptotic values.

Definition 2.14 (Natural families)**.** A holomorphic family $\{f_\lambda\}_{\lambda\in\Lambda}$ of meromorphic maps is called a *natural family* if there exists $\lambda_0 \in \Lambda$ and quasiconformal homeomorphisms $\varphi_\lambda, \psi_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ depending holomorphically on $\lambda \in \Lambda$ such that for all $\lambda \in \Lambda$, and

$$f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}.$$

The maps $\varphi_\lambda(z), \psi_\lambda(z)$ can be thought of as holomorphic motions of the Riemann sphere over Λ, with base point λ_0.

We will often use the following observation: if $\{f_\lambda\}_{\lambda\in\Lambda}$ is a natural family and $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$ for some $\lambda_0 \in \Lambda$, then we can choose any $\lambda_1 \in \Lambda$ as a new basepoint without changing f_λ, by replacing φ_λ with $\varphi_\lambda \circ \varphi_{\lambda_1}^{-1}$ and ψ_λ with $\psi_\lambda \circ \psi_{\lambda_1}^{-1}$.

We also note the following basic but important properties of natural families: if $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$, then

1. ψ_λ fixes ∞, since this is the only essential singularity of f_λ for all λ.
2. ψ_λ maps critical points of f_{λ_0} to critical points of f_λ, preserving multiplicites. In particular, all critical points move holomorphically over Λ and have constant multiplicity.
3. φ_λ maps critical values of f_{λ_0} to critical values of f_λ, and asymptotic values of f_{λ_0} to asymptotic values of f_λ. In particular, singular values move holomorphically over Λ.
4. Let $\mathcal{V}_\lambda$ denote the set of Picard exceptional values of f_λ. Then $\mathcal{V}_\lambda = \varphi_\lambda(\mathcal{V}_{\lambda_0})$. Indeed, $f_{\lambda_0}(z) = w$ iff $f_\lambda \circ \psi_\lambda(z) = \varphi_\lambda(w)$, hence since $\psi_\lambda, \varphi_\lambda$ are homeomorphisms, w has infinitely many preimages under f_{λ_0} iff $\varphi_\lambda(w)$ has infinitely many preimages under f_λ.

The following is a criterion for proving that a family of finite type meromorphic maps is natural.

Theorem 2.15 (Characterization of natural families, [ABF21, Theorem 2.6])**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a holomorphic family of finite type meromorphic maps, for which $S(f_\lambda)$ and $f_\lambda^{-1}(S(f_\lambda))$ both move holomorphically. Then for every $\lambda \in M$, there is a neighborhood V of λ such that $\{f_\lambda\}_{\lambda\in V}$ is a natural family.*

Meromorphic maps with asymptotic values may have a special type of cycles which play a special role in the dynamics of the map.

Definition 2.16 (Virtual cycle)**.** Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic map. A *virtual cycle* of length n is a finite, cyclically ordered set $z_0, z_1, \ldots, z_{n-1}$ such that for all i, either $z_i \in \mathbb{C}$ and $z_{i+1} = f(z_i)$, or $z_i = \infty$ and z_{i+1} is an asymptotic value for f, and at least one of the z_i is equal to ∞. If $z_i = \infty$ only for one value of i then we say that the virtual cycle has *minimal length n*.

If a virtual cycle remains after perturbation within the family, then it is called *persistent*.

Definition 2.17 (Persistent virtual cycle)**.** Let $\{f_\lambda\}_{\lambda\in M}$ be a holomorphic family of meromorphic maps, let $\lambda_0 \in M$ and assume that f_{λ_0} has a virtual cycle $z_0, \ldots, z_{n-1}$. If there exist holomorphic germs $\lambda \mapsto z_i(\lambda)$ defined near λ_0 in M such that

1. $z_i(\lambda_0) = z_i$
2. $z_i(\lambda) \equiv \infty$ if $z_i = \infty$
3. and $z_0(\lambda), \ldots, z_{n-1}(\lambda)$ is a virtual cycle for f_λ,

then we say that $z_0, \dots, z_{n-1}$ is a *persistent* virtual cycle.

In particular in a holomorphic family, if $v(\lambda_0)$ is an asymptotic value such that $f_{\lambda_0}^n(v(\lambda_0)) = \infty$ for some $n \geq 0$, then $(v(\lambda_0), f_{\lambda_0}(v(\lambda_0)), \dots, \infty)$ is a virtual cycle of minimal length $n+1$. (The case $n = 0$ corresponds to the situation where ∞ itself is an asymptotic value). This virtual cycle is persistent if and only if the singular relation $f_\lambda^n(v(\lambda)) = \infty$ is satisfied in all of M. If this is not the case, i.e. if a virtual cycle for λ_0 is non-persistent, we will say that λ_0 is a *virtual cycle parameter.*

Our next definition concerns the concept of activity or passivity of a singular value (see also [Lev81], [McM94]).

Definition 2.18 (Passive and active singular value)**.** Let $\{f_\lambda\}_{\lambda \in M}$ be a holomorphic family of meromorphic maps. Let $v(\lambda)$ be a singular value (or a critical point) of f_λ depending holomorphically on λ near some $\lambda_0 \in M$. We say that $v(\lambda)$ is *passive* at λ_0 if there exists a neighborhood V of λ_0 in M such that:

(1) either $f_\lambda^n(v(\lambda)) = \infty$ for all $\lambda \in V$; or
(2) the family $\{\lambda \mapsto f_\lambda^n(v(\lambda))\}_{n \in \mathbb{N}}$ is well-defined and normal on V.

We say that $v(\lambda)$ is *active* at λ_0 if it is not passive.

Recall that a family of meromorphic maps is said to be of *finite type* if $S(f_\lambda)$ is finite. For such natural families, the following are equivalent definitions of $J-$stability.

Theorem 2.19 (Characterizations of $\mathcal{J}$-stability, [ABF21, Theorem E])**.** *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of finite type meromorphic maps. Let $U \subset M$ be a simply connected domain of parameters. The following are equivalent:*

(1) *The Julia set moves holomorphically over U (i.e. f_λ is $\mathcal{J}$-stable for all $\lambda \in U$)*
(2) *Every singular value is passive on U.*
(3) *The maximal period of attracting cycles is bounded on U.*
(4) *The number of attracting cycles is constant in U.*
(5) *For all $\lambda \in U$, f_λ has no non-persistent parabolic cycles.*

In [ABF21, Proposition 5.5] it is shown that parameters with virtual cycles are dense. We will use this result several times.

Proposition 2.20 (Density of parameters with virtual cycles)**.** *Let $\{f_\lambda\}_{\in M}$ be a natural family of finite type meromorphic maps, and assume that there exists $\lambda_* \in M$ such that f_{λ_*} has at least one non-omitted pole. Assume that an asymptotic value is active at some $\lambda_0 \in M$. Then λ_0 can be approximated by virtual cycle parameters of arbitrarily large order.*

In [ABF24, Proposition 2.23] is proven a version of Proposition 2.20 which applies to meromorphic maps (even if they are not of finite type). We will use this in Section 6.

Misiurewicz parameters are also dense ([ABF21], Corollary 5.6). Recall that for a given λ, a singular value v_λ is Misiurewicz if there exists k such that $f_\lambda^k(v_\lambda) = p$ with p repelling periodic point, non-persistently in λ.

Proposition 2.21 (Density of Misiurewicz paramteres)**.** *Let $\{f_\lambda\}_{\in M}$ be a natural family of finite type meromorphic maps, and assume that there exists $\lambda_* \in M$ such that f_{λ_*} has at least one non-omitted pole. Assume that a singular value v_λ is active at some $\lambda_0 \in M$. Then λ_0 can be approximated by parameters for which v_λ is Misiurewicz.*

Finally, we will make use several times of the following Shooting Lemma ([ABF21, Proposition 2.13]).

Proposition 2.22 (Shooting Lemma). *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of meromorphic maps. Let $\lambda_0 \in M$ be such that a singular value v_λ satisfies $f^n_{\lambda_0}(v_{\lambda_0}) = \infty$, but this relation is not satisfied for all $\lambda \in M$. Let $\lambda \mapsto \gamma(\lambda)$ be a holomorphic map such that $\gamma(\lambda_0) \notin S(f_{\lambda_0})$. Then we can find λ' arbitrarily close to λ_0 such that $f^{n+1}_{\lambda'}(v_{\lambda'}) = \gamma(\lambda')$.*

2.4. **Tracts over asymptotic values.** In our proofs it will be important to control the size of *tracts* and how they move with respect to parameters.

Definition 2.23 (Tracts). Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic map, let $v \in \hat{\mathbb{C}}$ be an asymptotic value of f, and D be a punctured disk centered at v of radius $r > 0$. We say that a simply connected unbounded set T is a *logarithmic tract above* v if $f : T \to D$ is an infinite degree unbranched covering.

Notice that by decreasing the value of the radius r, we obtain a nested sequence of logarithmic tracts shrinking to infinity. We say that two logarithmic tracts (or sequences thereof) are different, if they are disjoint for sufficiently small values of r. The number of different logarithmic tracts lying over an asymptotic value v is the *multiplicity* of v.

If v is an isolated asymptotic value one can see that there is always a logarithmic tract above v (see e.g. [BE08]). In particular, if f is of finite type all of its asymptotic values have logarithmic tracts lying over them. For this reason, in this paper we will call them simply *tracts*.

The following properties of tracts for natural families will be used only in Section 6 for the proof of Theorem C.

Lemma 2.24 (Tracts move holomorphically). *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a natural family of meromorphic maps, where $\Lambda \subset \mathbb{C}$ is a simply connected domain. Let v_λ be an asymptotic value for f_λ, and assume that there exists a holomorphic motion $w_\lambda : D_{\lambda_0} \to D_\lambda$ of a Jordan domain D_{λ_0} such that for all $\lambda \in \Lambda$, $D_\lambda \cap S(f_\lambda) = \{v_\lambda\}$ and $w_\lambda(v_{\lambda_0}) = v_\lambda$. Let T_{λ_0} denote a tract above D_{λ_0} for f_{λ_0}. Then there is a holomorphic motion $m_\lambda : T_{\lambda_0} \to T_\lambda$ over M such that for all $\lambda \in \Lambda$, T_λ is a tract above D_λ for f_λ and*

$$f_\lambda \circ m_\lambda = w_\lambda \circ f_{\lambda_0}.$$

Proof. (See also the proof of Lemma 6.10.) We may write $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$, where $\varphi_\lambda, \psi_\lambda$ are holomorphic motions of $\hat{\mathbb{C}}$, with basepoint λ_0. Let $z_0 \in T_{\lambda_0}$. Consider the continuous map $H : \Lambda \to \mathbb{C}$ defined as $H(\lambda) = \varphi_\lambda^{-1} \circ w_\lambda \circ f_{\lambda_0}(z_0)$.

By the lifting property for covering maps, there exists a unique lift $\gamma_{z_0}(\lambda)$ of H by f_{λ_0} such that $\gamma_{z_0}(\lambda_0) = z_0$.

$$\begin{tikzcd}
& \mathbb{C}\setminus f_{\lambda_0}^{-1}(S(f_{\lambda_0})) \arrow[d, "f_{\lambda_0}"] \\
\Lambda \arrow[ur, "\gamma_{z_0}"] \arrow[r, "H"] & \mathbb{C}\setminus S(f_{\lambda_0}).
\end{tikzcd}$$

Indeed, since Λ is simply connected, we only need to check that $v_{\lambda_0} \notin H(\Lambda)$, so that f_{λ_0} is a covering over $H(\Lambda)$. This is true because by assumption $\varphi_\lambda^{-1} \circ w_\lambda(v_{\lambda_0}) = v_{\lambda_0}$, $\varphi_\lambda^{-1} \circ w_\lambda$ is injective, and $f_{\lambda_0}(z_0) \neq v_{\lambda_0}$. By definition of lifting, $f_{\lambda_0} \circ \gamma_{z_0}(\lambda) = H(\lambda) = \varphi_\lambda^{-1} \circ w_\lambda \circ f_{\lambda_0}(z_0)$.

We then let $m_\lambda(z_0) := \psi_\lambda \circ \gamma_{z_0}(\lambda)$. By construction, we have

$$f_\lambda \circ m_\lambda(z_0) = f_\lambda \circ \psi_\lambda \circ \gamma_{z_0}(\lambda) = \varphi_\lambda \circ f_{\lambda_0} \circ \gamma_{z_0}(\lambda) = \varphi_\lambda \circ H = w_\lambda \circ f_{\lambda_0}(z_0) \tag{2.1}$$

and $m_{\lambda_0}(z_0) = z_0$ since $\psi_{\lambda_0} = \mathrm{id}$. By construction, $\gamma_{z_0}(\lambda) \notin f_{\lambda_0}^{-1}(S(f_{\lambda_0}))$, for all $\lambda \in \Lambda$. In particular, it is never a critical point of f_{λ_0}. Since ψ_λ maps critical points of f_{λ_0} to critical points of f_λ, we therefore have $f'_\lambda(m_\lambda(z_0)) \neq 0$ for all $\lambda \in \Lambda$. By the Implicit Function Theorem, we have that $\lambda \mapsto m_\lambda(z_0)$ is holomorphic in Λ. By the uniqueness of the lifting property, the map $z \mapsto m_\lambda(z)$ is injective for all $\lambda \in \Lambda$. Thus, $m_\lambda : \Lambda \times T_{\lambda_0} \to \hat{\mathbb{C}}$ is indeed a holomorphic motion, and it follows from (2.1) that $T_\lambda := m_\lambda(T_{\lambda_0})$ is a tract for f_λ above D_λ. □

Lemma 2.25 (Tracts above small disks are small). *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a natural family of meromorphic maps and let v_λ be an isolated asymptotic value. For any $\delta > 0$, we denote by $T_\lambda(\delta)$ a nested family of tracts above $\mathbb{D}(v_\lambda, \delta)$, i.e. $T_\lambda(\delta_1) \subset T_\lambda(\delta_2)$ if $0 < \delta_1 \leq \delta_2$. Let $\lambda_0 \in \Lambda$. For every $r > 0$, there exists $\delta > 0$ and $\eta > 0$ such that for any $\lambda \in \mathbb{B}(\lambda_0, \eta)$, any tract $T_\lambda(\delta)$ above $\mathbb{D}(v_\lambda, \delta)$ for f_λ is compactly contained in $\mathbb{D}(\infty, r)$.*

Proof. We first prove the statement for $f = f_{\lambda_0}$ and $v = v_{\lambda_0}$. Suppose for a contradiction that there exists $r > 0$, a decreasing sequence $\delta_n \to 0$, a sequence of tracts $T_n := T_{\lambda_0}(\delta_n)$ over $\mathbb{D}(v, \delta_n)$ and points $z_n \in T_n$ such that for all n, the spherical distance satisfies $d(z_n, \infty) \geq r$. Up to extracting a subsequence, we may assume that $z_n \to z \in \mathbb{C}$. Moreover, since the tracts are nested, we have $z \in \overline{T_n} \subset \overline{T_0}$ for all $n \in \mathbb{N}$, hence $z \in \overline{T_0} \cap \mathbb{C}$. Since $f(z_n) \in \mathbb{D}(v, \delta_n)$, by continuity $f(z) = v$, which gives a contradiction because $f(\overline{T_0} \cap \mathbb{C}) = \overline{\mathbb{D}(v, \delta_0)} \setminus \{v\}$.

The statement for λ in a neighborhood of λ_0 then follows from the continuity of the tracts with respect to the parameters. □

3. Tangent-like maps

In this section we introduce tangent-like maps, filled Julia sets, and the notion of hybrid equivalence. We then show that the filled Julia set is connected if and only if it contains the singular value.

3.1. First definitions: tangent-like maps and hybrid equivalence.

The definition of tangent-like maps (TL for short) first appeared in [GK08].

Definition 3.1 (Tangent-like map). A tangent-like map is a tuple (f, U, V, u, v) such that

(1) $U, V \subset \hat{\mathbb{C}}$ are Jordan domains, and $\overline{U} \subset V$
(2) $u \in \partial U$, $v \in V$
(3) $f : \overline{U} \setminus \{u\} \to \overline{V} \setminus \{v\}$ is a covering map
(4) $f : U \to V$ is holomorphic
(5) ∂U is a quasicircle and ∂V is real-analytic.

See Figure 1. Since U is simply connected and $V \setminus \{v\} \cong \mathbb{D}^*$, it follows from the classification of coverings of $\mathbb{D}^*$ that $f : U \to V \setminus \{v\}$ is equivalent to the exponential map, so u is an essential singularity (in the sense that f cannot be continuously extended at u), v is an asymptotic value, and f has infinite degree. Note that, up to restricting V slightly, we can always assume that ∂V is real-analytic; this will imply that ∂U is analytic except possibly at u.

Definition 3.2 (Filled Julia set). Let (f, U, V, u, v) be a tangent-like map. The filled Julia set is

$$K(f) := \bigcap_{n\geq 0} \overline{f^{-n}(U)}.$$

The Julia set is $J(f) := \partial K(f)$.

Remark 3.3. By definition, we always have $u \in J(f)$. Indeed, f is equivalent to an exponential map where u is an essential singularity, hence for any $z \in V$, $u \in \overline{f^{-1}(\{z\})}$, and $u \in \partial U$ so $u \in \partial K(f)$.

The fact that $f : U \to V$ is neither surjective nor proper, and does not extend continuously at u, calls for a little caution in the proof of the next proposition.

Proposition 3.4 (Complete invariance of $K(f)$)**.** *The set $K(f)$ is completely invariant, in the sense that $f^{-1}(K(f)) = K(f) \setminus \{u\}$. Moreover, we have*

$$K(f) = \left(\bigcup_{n\geq 0} f^{-n}(\{u\})\right) \sqcup \left(\bigcap_{n\geq 0} f^{-n}(U)\right).$$

Of course, since the map f is not defined at u, u cannot belong to $f^{-1}(K(f))$.

Proof. Let us first prove that $f^{-1}(K(f)) \subset K(f) \setminus \{u\}$.

Let $z \in f^{-1}(K(f)) \subset \overline{U} \setminus \{u\}$, so that $f(z) \in K(f)$. Then by definition, for every $n \in \mathbb{N}$, there exists a sequence $y_{k,n}$ such that $\lim_{k\to+\infty} y_{k,n} = f(z)$, and $f^n(y_{k,n}) \in U$. Let g be an inverse branch of f mapping $f(z)$ to z (it exists because $f : U \to V \setminus \{v\}$ is a covering map). For k large enough, $y_{k,n}$ is in the domain of g, and we let $x_{k,n} := g(y_{k,n})$. Then $\lim_{k\to+\infty} x_{k,n} = z$, and $f^n(x_{k,n}) = f^{n-1}(y_{k,n}) \in U$. Therefore $z \in \bigcap_{n\geq 0} \overline{f^{-n}(U)} = K(f)$.

Conversely, let $z \in K(f)\setminus\{u\}$. By definition, for every $n \geq 1$ there exists a sequence $x_{k,n}$ such that $\lim_{k\to+\infty} x_{k,n} = z$, and $f^n(x_{k,n}) \in U$. Let $y_{k,n} := f(x_{k,n})$: then $\lim_{k\to+\infty} y_{k,n} = f(z)$, and $f^{n-1}(y_{k,n}) \in U$. This proves that $f(z) \in K(f)$, and therefore that

$$f^{-1}(K(f)) \supset K(f) \setminus \{u\}$$

as required.

We now turn to the second assertion in the Lemma. Let $z \in K(f)$, such that $z \notin \bigcap_{n\geq 0} f^{-n}(U)$. Then again, by definition of $K(f)$, for every $n \geq 0$ there exists a sequence $x_{k,n}$ such that $\lim_{k\to+\infty} x_{k,n} = z$, and $f^n(x_{k,n}) \in U$. Morever, there exists $n_0 \in \mathbb{N}$ such that $f^{n_0}(z) \notin U$. Let us prove that $f^{n_0}(z) = u$. First observe that $f^{n_0}(z) \in \partial U$. Indeed, otherwise we would have $f^{n_0}(z) \in V \setminus \overline{U}$ which is open, so for k large enough we would have $f^{n_0}(x_{k,n_0}) \in V \setminus \overline{U}$, a contradiction. Next, if $f^{n_0}(z) \in \partial U \setminus \{u\}$, then we would have $f^{n_0+1}(z) \in \partial V$, and so by the same reasonning, $f^{n_0+1}(x_{k,n_0+1}) \notin U$ for k large enough, again a contradiction. Therefore $f^{n_0}(z) = u$.

This proves that $K(f) \subset \left(\bigcup_{n\geq 0} f^{-n}(\{u\})\right) \sqcup \left(\bigcap_{n\geq 0} f^{-n}(U)\right)$.

Finally, the inclusion $\left(\bigcup_{n\geq 0} f^{-n}(\{u\})\right) \sqcup \left(\bigcap_{n\geq 0} f^{-n}(U)\right) \subset K(f)$ follows from the relation $f^{-1}(K(f)) = K(f) \setminus \{u\}$ and the fact that $u \in K(f)$.

□

Remark 3.5. In particular, there is exactly a countable set of points in $K(f)$ with finite orbit (which act as prepoles), unless $u = v$, in which case there are none.

The concept of hybrid equivalence can be defined in this context analogously as for polynomial-like maps.

Definition 3.6 (Hybrid equivalent tangent-like maps)**.** Two tangent-like maps $(f_1, U_1, V_1, u_1, v_1)$ and $(f_2, U_2, V_2, u_2, v_2)$ are hybrid equivalent if there exists a quasiconformal homeomorphism $\varphi : V_1 \to V_2$ such that for all $z \in U_1$,

$$\psi \circ f_1(z) = f_2 \circ \varphi(z)$$

and $\overline{\partial}\psi = 0$ a.e. on $K(f_1)$.

3.2. **Connectedness of** $K(f)$**.** As in the quadratic-like case, the topology of $K(f)$ is determined by the dynamics of the singular value.

Proposition 3.7. *Let (f, U, V, u, v) be a tangent-like map. Then $K(f)$ is connected if and only if $v \in K(f)$.*

Remark. In fact, if $v \notin K(f)$, then $K(f)$ is totally disconnected. This will follow from the Straightening theorem and the analogue result for the model family proven in [Ga11, Lemma 3.3], although it could probably be proven directly.

Proof. Let us first assume that $v \in K(f)$ and let us prove that $K(f)$ is connected. Either for all $n \in \mathbb{N}$, $f^n(v) \neq u$, or there exists $n_0 \in \mathbb{N}$ such that $f^{n_0}(v) = u$. In the first case, by Lemma 3.4, $v \in f^{-n}(U)$ for all $n \in \mathbb{N}$. We prove by induction on n that $f^{-n}(U)$ is a simply connected domain. For $n = 0$, U is simply connected by definition; now let $n \in \mathbb{N}$ and assume that $f^{-n}(U)$ is a simply connected domain. Then $f^{-(n+1)}(U)$ is a logarithmic tract above $f^{-n}(U)$ (since $v \in f^{-n}(U)$). Therefore, $f^{-(n+1)}(U)$ is indeed simply connected. The closure of a connected set is connected, and the decreasing intersection of connected compact sets is connected. Therefore $K(f) = \bigcap_{n \geq 0} \overline{f^{-n}(U)}$ is connected.

Let us now assume that there exists $n_0 \in \mathbb{N}$ such that $f^{n_0}(v) = u$. By the same argument as above, we have that for all $n \leq n_0$, $f^{-n}(U)$ is simply connected. But now $f^{-n_0}(U)$ is simply connected, and v belongs to the boundary of $f^{-n_0}(U)$; so $f^{-n_0-1}(U)$ is a countable union of disjoint simply connected sets, which also do not contain v. Moreover, the closure of each connected component of $f^{-n_0-1}(U)$ contains u. By a quick induction on n, we have that for all $n > n_0$, $f^{-n}(U)$ is a disjoint union of countably many simply connected domains, each having u in their closure. In particular, even though $f^{-n}(U)$ is disconnected, $\overline{f^{-n}(U)}$ is connected. Since the decreasing intersection of a sequence of connected sets is connected, $K(f)$ is again connected.

Conversely, assume that $v \notin K(f)$. Then there exists $n_0 \in \mathbb{N}$ (minimal) such that $v \notin f^{-n_0}(U)$. By the same reasoning as above, $f^{-n_0}(U)$ is simply connected, and every connected component of $f^{-(n_0+1)}(U)$ maps one to one to $f^{-n_0}(U)$; however in this case, v is not in the boundary of $f^{-n_0}(U)$, so the aforementioned connected components do not contain u in their boundary . In particular, $\overline{f^{-(n_0+1)}(U)}$ is disconnected. Since each component of $f^{-(n_0+1)}(U)$ (open) contains some Julia set (closed) (by backward invariance) and since $J \subset f^{-(n_0+1)}(U)$, the Julia set is disconnected. □

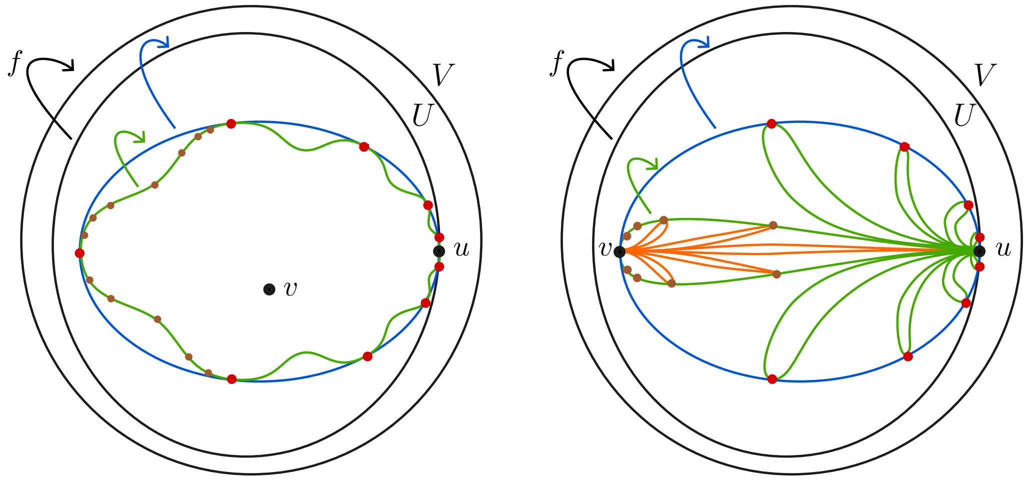

FIGURE 6. Sketch of the proof of Proposition 3.7, in the case when $v \in K(f)$. In blue, $f^{-1}(\partial U)$, an unbounded Jordan curve which contains all poles (preimages of u) of f (red dots). In green, $f^{-2}(\partial U)$, which contains all prepoles of order one or two of f (red and brown dots). Left: $f^n(u) \neq v$ for all $n \geq 0$, hence $f^{-n}(\partial U)$ is a single (unbounded) Jordan curve. Right: $f(v) = u$ ($n_0 = 1$). Hence $f^{-2}(\partial U)$ consists of infinitely many unbounded curves mapping one-to-one to ∂U. In orange, $f^{-3}(\partial U)$, illustrating the pattern that repeats inside every preimage of U.

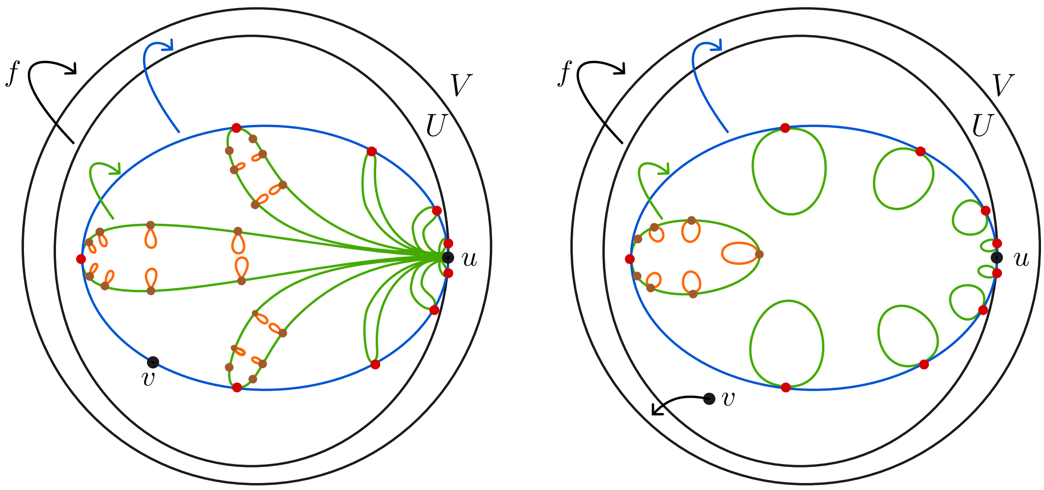

FIGURE 7. Sketch of the proof of Proposition 3.7, in the case $n_0 = 1$, that is when $f^2(v) \notin \overline{U}$. Left: $f(v) \in \overline{U}$. Right: $f(v) \notin \overline{U}$.

4. The model family

In analogy to quadratic polynomials, viewed as the slice of Rat_2 (degree two rational maps) for which one of the two critical points is superattracting, we consider here the one-dimensional slice of $\mathbb{F}_2$ (meromorphic maps with exactly two ommitted asymptotic values, known as *generalized tangent maps*, (see the discussion in [FK21, Appendix] and also [DK89, KK97, EG09]) with an attracting fixed point of constant multiplier. For simplicity purposes, we fix the value of the multiplier to be 1/8, although it can be shown that this choice is irrelevant.

Maps in this slice are of the form

$$T_\alpha(z) := \frac{e^{(\alpha-1)z/8} - 1}{\alpha e^{(\alpha-1)z/8} - 1}, \tag{4.1}$$

where $\alpha \in \mathbb{C} \setminus \{1\}$.

In a way, $\{T_\alpha\}_\alpha$ is the simplest family whose members are naturally tangent-like maps, in the same way that quadratic polynomials are the simplest functions which are naturally quadratic-like maps.

In this section, we shall study the general properties of the maps T_α, both in dynamical and in parameter plane. In the latter, we shall introduce the *Tandelbrot set*, an infinite analogue of the Mandelbrot set.

Remark. The slice(s) that we consider in this paper, up to taking a different normalization, have been extensively studied in [CJK22, CJK23, GK08, Ga11], where a detailed combinatorial description of both dynamical and parameter plane is given. There is little overlap with the results in the present paper; we indicate explicitely where such overlap occurs.

4.1. **First properties.** Let T_α be defined as in (4.1). For $\alpha \in \mathbb{C} \setminus \{1\}$, the map T_α has two asymptotic values, 1 and $\frac{1}{\alpha}$. It also fixes 0 with multiplier equal to $\frac{1}{8}$, hence at least one of the two asymptotic values must always belong to the basin of 0. In particular, their activity loci are disjoint. Finally, observe that for $\alpha = 0$, $T_0(z) = 1 - e^{-z/8}$ is an exponential map of disjoint type (i.e. the Fatou set of T_0 is connected and consists of a completely invariant basin of attraction of a fixed point). This is the only parameter value for which T_α is an entire map.

We shall see that when $\alpha \in \overline{\mathbb{D}} \setminus \{1\}$, the asymptotic value 1 is always contained in the basin of attraction of 0, so that $1/\alpha$ acts as a *free* asymptotic value.

Lemma 4.1 ($1/\alpha$ is free if α is small)**.** *For all* $\alpha \in \overline{\mathbb{D}} \setminus \{1\}$, *we have* $T_\alpha(\mathbb{D}(0,2)) \subset \mathbb{D} \Subset \mathbb{D}(0,2)$. *Therefore,* $\overline{\mathbb{D}}(0,2)$ *(and in particular the asymptotic value* 1*) lie in the basin of attraction of* $z = 0$.

Proof. Let $\alpha \in \overline{\mathbb{D}} \setminus \{1\}$. Then T_α has no singular values in $\mathbb{D}$, since its singular values are 1 and $\frac{1}{\alpha}$. Therefore, there exists a well-defined univalent inverse branch g of T_α^{-1} fixing 0 and defined on $\mathbb{D}$. We have $g'(0) = 8$; thus, by Koebe's distortion theorem, $g(\mathbb{D}) \supset \mathbb{D}(0,2)$. In particular, this implies that $T_\alpha(\mathbb{D}(0,2)) \subset \mathbb{D}$ and therefore that $T_\alpha(\overline{\mathbb{D}}(0,2)) \subset \overline{\mathbb{D}} \Subset \mathbb{D}(0,2)$. $\square$

Next, we prove that our family is characterized by its dynamical features: up to an affine conjugacy, any meromorphic map with exactly two singular values and a fixed point of multiplier $\frac{1}{8}$ belongs to the model family. Additionnally, the parameter space is invariant under the involution $\alpha \mapsto \frac{1}{\alpha}$.

Lemma 4.2 (Affine conjugacy classes)**.** *Any meromorphic transcendental map with exactly two singular values and a fixed point with multiplier equal to* $\frac{1}{8}$ *is affinely conjugate to a map* T_α. *Moreover, if* $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0,1\}$, *then* T_{α_1} *is affinely conjugate to* T_{α_2} *if and only if* $\alpha_2 = \frac{1}{\alpha_1}$ *or* $\alpha_2 = \alpha_1$.

Proof. Let $\mu, \lambda \in \mathbb{C}$ denote the two distinct singular values and $M(z) := \frac{z-\mu}{z-\lambda}$. Then, the map $M \circ f : \mathbb{C} \setminus \{f^{-1}(\mu), f^{-1}(\lambda)\} \longrightarrow \mathbb{C} \setminus \{0, \infty\}$ is a covering. Coverings of $\mathbb{C}^*$ are either equivalent to z^d and then they have finite degree, or to the exponential map in which case the domain is simply connected.

In this case, since the degree is infinite, the domain is simply connected, i.e. 0 and ∞ are ommitted by $M \circ f$, and $M \circ f$ is a universal covering of $\mathbb{C} \setminus \{0, \infty\}$. By uniqueness of universal coverings, $f = M^{-1}(\exp(L(z))$, where L is a conformal map of $\mathbb{C}$ and hence afine.

Thus $f(z) = \frac{\lambda e^{az+b}-\mu}{e^{az+b}-1}$, for some $a \in \mathbb{C}^*$, $b \in \mathbb{C}$. Let A denote an affine map sending μ to 1 and the fixed point of f with multiplier 1/8 to 0. It is then straightforward to check that $A \circ f \circ A^{-1} = T_\alpha$, where $\alpha = 1/\lambda$.

Finally, if $\lambda = \infty$ or $\mu = \infty$, the same procedure results in $T_0(z)$.

Let us now prove that for all $\alpha \in \mathbb{C} \setminus \{0, 1\}$, T_α is linearly conjugated to $T_{1/\alpha}$ by the linear map $L(z) := \alpha z$. Indeed,

$$L \circ T_\alpha \circ L^{-1}(z) = \alpha \frac{e^{z(\alpha-1)/(8\alpha)} - 1}{\alpha e^{z(\alpha-1)/(8\alpha)} - 1} = \frac{e^{(1-1/\alpha)z/8} - 1}{e^{(1-1/\alpha)z/8} - \frac{1}{\alpha}} = \frac{e^{(1/\alpha-1)z/8} - 1}{\frac{1}{\alpha} e^{(1/\alpha-1)z/8} - 1} = T_{1/\alpha}(z).$$

Finally, let us prove that if T_{α_1} is affinely conjugated to T_{α_2} , then $\alpha_2 = \frac{1}{\alpha_1}$ or $\alpha_1 = \alpha_2$. Let A denote the affine conjugacy such that $A \circ T_{\alpha_1} \circ A^{-1} = T_{\alpha_2}$ and set $A(z) := az + b$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$. Then A must map $S(T_{\alpha_1}) = \{1, \frac{1}{\alpha_1}\}$ to $S(T_{\alpha_2}) = \{1, \frac{1}{\alpha_2}\}$.

Suppose first that $A(0) = 0$ and hence $b = 0$. If $A(1) = 1$, then A fixes two points and hence it is the identity. If $A(1) = 1/\alpha_2$, then $a = 1/\alpha_2$. But then, $A(1/\alpha_1) = 1$ and hence $a = \alpha_1$. It follows then that $\alpha_1 = 1/\alpha_2$.

Nevertheless, it could also happen that $A(0) = z_2 \neq 0$, where z_2 is a fixed point of T_{α_2} with multiplier 1/8. (This situation could only happen for a countable number of parameters, for which the Fatou set consists of two completely invariant basins which are Jordan domains, see Remark 4.3.) More precisely, for $\alpha \in \mathbb{C} \setminus \{0, 1\}$, one can check that the fixed points of T_α must satisfy

$$e^{\frac{(\alpha-1)z}{8}} = \frac{z-1}{\alpha z - 1}, \tag{4.2}$$

while points of derivative 1/8 satisfy

$$e^{\frac{(\alpha-1)z}{8}} \in \{1, 1/\alpha^2\}. \tag{4.3}$$

From the two equations together, it follows that there is only one possible nonzero fixed point with multiplier 1/8, and that is $z_\alpha = 1 + \frac{1}{\alpha}$. Thus, an affine conjugacy switching the basin of 0 and the basin of z_{α_2} must satisfy $A(0) = 1 + \frac{1}{\alpha_2}$ and $A(1 + \frac{1}{\alpha_1}) = 0$ and hence has the form

$$A(z) = -\frac{\alpha_1(1+\alpha_2)}{\alpha_2(1+\alpha_1)} z + 1 + \frac{1}{\alpha_2}.$$

On the other hand, A must map $S(T_{\alpha_1}) = \{1, \frac{1}{\alpha_1}\}$ to $S(T_{\alpha_2}) = \{1, \frac{1}{\alpha_2}\}$. From the expression of A, one can check that if $A(1) = 1$ then $\alpha_1 = \alpha_2$ while, if $A(1) = 1/\alpha_2$, then $\alpha_2 = 1/\alpha_1$, as we wanted to show. □

Remark 4.3. (Maps with a symmetry switching the basins) If α is a parameter satisfying

$$e^{\frac{\alpha^2-1}{8\alpha}} = 1/\alpha^2,$$

then $z_\alpha = 1 + 1/\alpha$ is a fixed point of multiplier 1/8. The affine conjugacy $A(z) = -z + 1 + 1/\alpha$ conjugates T_α to itself switching the two attracting basins. Composing with $z \mapsto \alpha z$, we obtain an affine conjugacy to $T_{1/\alpha}$ switching the two basins and mapping 1 to itself. Examples of filled Julia sets for these parameter values can be seen in Figure 8.

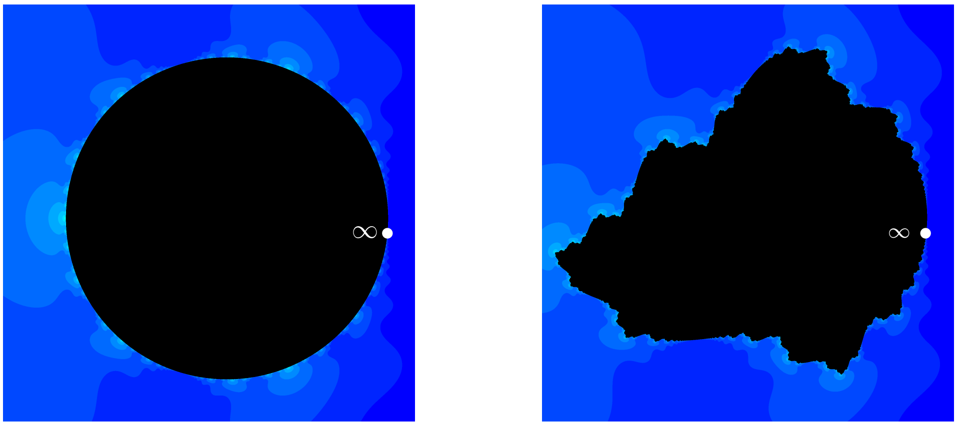

FIGURE 8. Filled Julia sets of T_α for $\alpha = -0.01484108...$ (left) and $\alpha = -0.00801734... + 0.00675639...i$ (right).

Next we show that for $\alpha \in \mathbb{D}$, T_α is a tangent-like map when seen from the complement of the basin of 0 (see Figure 9).

Lemma 4.4 (The maps T_α are tangent-like)**.** *For all $\frac{1}{2} \leq r < 1$, for all $\alpha \in \mathbb{D}(0, r)$, the map $T_\alpha : \hat{\mathbb{C}} \setminus T_\alpha^{-1}(\mathbb{D}(0, \frac{1}{r})) \to \hat{\mathbb{C}} \setminus \mathbb{D}(0, \frac{1}{r})$ is TL, with $u = \infty$ and $v = \frac{1}{\alpha}$.*

Proof. Let $\frac{1}{2} \leq r < 1$ and $\alpha \in \mathbb{D}(0, r) \subset \mathbb{D}$. Let $V := \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}(0, \frac{1}{r})$ and $U := T_\alpha^{-1}(V)$, and let $u := \infty$ and $v := \frac{1}{\alpha}$. Since $|\alpha| < r < 1$, V contains $v = \frac{1}{\alpha}$ but not 1. Moreover, V is a Jordan domain, whose boundary is real-analytic (in particular, ∂V is a quasicircle). The map $T_\alpha : U \to V \setminus \{v\}$ is a universal cover, hence U is simply connected. From the expression of T_α, it is also clear that ∂U is a real-analytic Jordan curve, in particular a quasicircle. Finally, by Lemma 4.1 and the fact that $r > \frac{1}{2}$, we have $T_\alpha(\mathbb{D}(0, \frac{1}{r})) \subset \mathbb{D} \Subset \mathbb{D}(0, \frac{1}{r})$, from which it follows that $U \Subset V$. $\square$

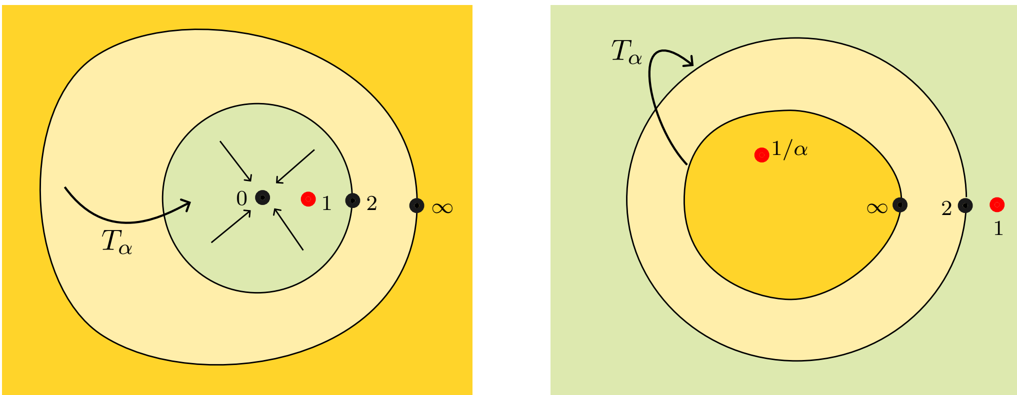

FIGURE 9. The Tangent-like restriction of the model family, seen from the complement of the basin of attraction of 0.

As it is the case for polynomials and polynomial-like maps, the choice of U and V is not really relevant and produce the same filled Julia set, which in fact is the whole Julia set of the model map, in this case T_α.

Proposition 4.5 (Choices of U and V)**.** *For any $\alpha \neq 1$, for any TL restriction $T_\alpha : U \to V$ with $u = \infty$ and $v = \frac{1}{\alpha}$:*

(1) the filled Julia set $K(T_\alpha)$ is $\hat{\mathbb{C}} \setminus \Omega_\alpha$, where Ω_α is the attracting basin of 0.
(2) the Julia set of T_α is also the Julia set of its TL restriction.

In particular, Julia sets and filled Julia sets of the TL restriction do not depend on the choice of U and V.

Proof. Let us first prove that $K(T_\alpha) = \hat{\mathbb{C}} \setminus \Omega_\alpha$. Indeed, observe that since $U \Subset V$, we have $\hat{\mathbb{C}} \setminus \overline{V} \Subset \hat{\mathbb{C}} \setminus \overline{U}$; and T_α maps $\hat{\mathbb{C}} \setminus \overline{U}$ into $\hat{\mathbb{C}} \setminus \overline{V}$. Therefore, the map f has a unique attracting fixed point $\hat{\mathbb{C}} \setminus \overline{V}$, which must be the attracting fixed point 0; and $\hat{\mathbb{C}} \setminus \overline{U} \subset \Omega_\alpha$. But by definition, if $z \notin K(T_\alpha)$, then there exists $n \in \mathbb{N}$ such that $T_\alpha^n(z) \in \hat{\mathbb{C}} \setminus \overline{U}$, so $z \in \Omega_\alpha$. This proves $\hat{\mathbb{C}} \setminus K(T_\alpha) \subset \Omega_\alpha$. Moreover, $\hat{\mathbb{C}} \setminus K(T_\alpha)$ is backward invariant, therefore, $\hat{\mathbb{C}} \setminus K(T_\alpha) = \Omega_\alpha$.

The second claim follows from the fact that the Julia set of T_α is $\partial \Omega_\alpha$. □

In particular, the general properties in Section 3 apply to maps T_α in the model family, for $\alpha \in \mathbb{D}$ and, by switching α and $1/\alpha$, for all $\alpha \in \mathbb{C} \setminus \overline{\mathbb{D}}$.

4.2. **Conformal conjugacy on the attracting basin of** 0. In this section we show that every tangent-like map T_α with connected filled Julia set is conformally conjugate on its attracting basin Ω_0 to the model map $g(z) = C \tan(\pi z) : \mathbb{H} \to \mathbb{H} \setminus \{Ci\}$, where C is a constant independent of α. In particular, all such tangent maps T_α are conformally conjugate to each other on the attracting basins of zero. This is an analogue of Böttcher's Theorem, with the model map z^d replaced by the map g. In fact this is more general, since it applies to completely invariant basins of attraction of a fixed point with given multiplier, containing exactly one asymptotic value and no other singular values. The constant C depends only on the multiplier. This was essentially proved already in [ERS20, Lemma 5.2].

Proposition 4.6 (Conjugacy on the basin of 0)**.** *Let $\alpha \in \mathbb{D}(0, 1/2)$ such that $K(T_\alpha)$ is connected, and let Ω_α be its attracting basin of 0. Then T_α is conformally conjugate on Ω_α to the map $g : \mathbb{H} \to \mathbb{H} \setminus \{Ci\}$ defined as*

$$g(z) = C \tan(\pi z),$$

where $C > 0$ is a constant independent of α. The conformal conjugacy is given by the Riemann map $\varphi_\alpha : \mathbb{H} \to \Omega_\alpha$, normalized so that $\varphi_\alpha^{-1}(1) = Ci$ and $p := \varphi_\alpha^{-1}(0) \in (0, Ci)$. Consequently, any two tangent-like maps satisfying the hypothesis of this proposition are conformally conjugate on their basins of attraction of 0.

Proof. Let $\widetilde{\varphi}_\alpha : \mathbb{H} \to \Omega_\alpha$ be the unique Riemann map satisfying $\widetilde{\varphi}_\alpha(i) = 1$ and $\widetilde{p} := \widetilde{\varphi}_\alpha^{-1}(0) \in (0, i)$. Then

$$\widetilde{g} := \widetilde{\varphi}_\alpha^{-1} \circ T_\alpha \circ \widetilde{\varphi}_\alpha$$

is a universal cover from $\mathbb{H} \to \mathbb{H} \setminus \{i\}$ which fixes a unique point $\widetilde{p}$ with multiplier $1/8$. A priori, $\widetilde{p}$ and $\widetilde{g}$ depend on α: let us show that this is not the case. Consider the Moebius map

$$M(z) = \frac{i(1-z)}{z+1} \tag{4.4}$$

mapping $\mathbb{D}$ to $\mathbb{H}$. Since $M(-1) = \infty$, $M(0) = i$, and $M(1) = 0$, we have that $M([-1, 1]) = i\mathbb{R}_+$, hence $p_* := M^{-1}(\widetilde{p}) \in [0, 1]$. Since M does not depend on α, if we show that p_* does not depend on α, we will obtain that $\widetilde{p}$ also does not depend on α. Observe that $|M'(p_*)| = \frac{2}{(p_*+1)^2}$, so

$$\rho_{\mathbb{H}\setminus\{i\}}(\widetilde{p}) = \frac{1}{|M'(p_*)|}\rho_{\mathbb{D}\setminus\{0\}}(p_*) = \frac{(p_*+1)^2}{2}\frac{1}{p_* \ln(p_*^{-1})}.$$

Imposing that $\widetilde{p}$ has multiplier $1/8$ and using that $\rho_{\mathbb{H}}(\widetilde{p}) = \frac{1}{\operatorname{Im}\,\widetilde{p}}$ and that $\widetilde{p} = \frac{i(1-p_*)}{1+p_*}$ we obtain the equation

$$\frac{1}{8} = |\widetilde{g}'(\widetilde{p})| = \frac{\rho_{\mathbb{H}}(\widetilde{p})}{\rho_{\mathbb{H}\setminus\{i\}}(\widetilde{p})} = \frac{2p_* \ln(p_*^{-1})}{1 - p_*^2}.$$

One can check that the function $x \mapsto \frac{2x \log x^{-1}}{1-x^2}$ is strictly increasing from $(0, 1)$ to $(0, 1)$, so that this relation uniquely determines p_* and hence $\widetilde{p}$.

Write now $\widetilde{g}$ as $\widetilde{g} = M \circ f$, where $f : \mathbb{H} \to \mathbb{D}_*$ is a universal covering mapping $\widetilde{p}$ to p_*. The map f is well defined because M is conformal. Since $e^{iz} : \mathbb{H} \to \mathbb{D}_*$ is also a universal covering and the automorphisms of $\mathbb{H}$ fixing infinity are of the form $az + b$ with $a > 0, b \in \mathbb{R}$ we have that $f(z) = e^{i(az+b)}$ (with $a > 0, b \in \mathbb{R}$ to be determined). It follows from this explicit expression that f extends to $\mathbb{R}$ and maps vertical lines in $\mathbb{H}$ to radii in $\mathbb{D}_*$. So, since $\widetilde{p} \in (0, i)$ and $p_* = f(\widetilde{p}) \in (0, 1)$, we have that $f(i\mathbb{R}_+) = (0, 1)$, and hence that $f(0) = 1$. It follows that $b = 0$ and $f(z) = e^{iaz}$ with $a > 0$ and that

$$\widetilde{g} = M \circ f = \frac{i(1 - e^{iaz})}{e^{iaz} + 1} = -\frac{i(e^{iaz/2} - e^{-iaz/2})}{e^{iaz} + e^{-iaz/2}} = \tan(\frac{az}{2}) =: \tan(C\pi z),$$

with $C = \frac{a}{2\pi} > 0$ yet to be determined.

Imposing that $\widetilde{p} \in (0, i)$ is a fixed point of $\widetilde{g}$, we obtain that

$$C = \frac{\operatorname{arctanh}(|\widetilde{p}|)}{\pi|\widetilde{p}|}.$$

Since $\widetilde{p}$ is independent of α, this determines C uniquely. By conjugating $\widetilde{g}$ with $L : z \mapsto Cz$ we get that

$$g(z) := (L \circ \widetilde{g} \circ L^{-1}) = C \tan(\pi z),$$

and that the conjugacy between T_α and g is given by $\varphi_\alpha := L \circ \widetilde{\varphi}_\alpha$ which satisfies the claim of the lemma. ☐

The conjugacy φ_α normalized as above allows us to mark some special points in $J(T_\alpha)$. In particular, there is a unique labeling for the poles independent of α, as long as $\alpha \neq 0$. See Figure 10.

Lemma 4.7 (Labelling of poles and fixed point)**.** *Let $\alpha \in \mathbb{D}(0, 1/2) \setminus \{0\}$ and φ_α, g be defined as in Lemma 4.6. Then*

$$\lim_{t\to\infty} \varphi_\alpha(it) = \infty. \tag{4.5}$$

Consequently, all poles of T_α are accessible from Ω_α and can be consistently labelled as

$$p_k := p_k(\alpha) := \lim_{t\to 0} \varphi_\alpha\left(\frac{1}{2} + k + i\,t\right), k \in \mathbb{Z}. \tag{4.6}$$

Moreover, we can consistently define a special fixed point, which we will call the β_0-fixed point*, as*

$$\beta_0 := \beta_0(\alpha) := \lim_{t\to 0} \varphi_\alpha(i\,t).$$

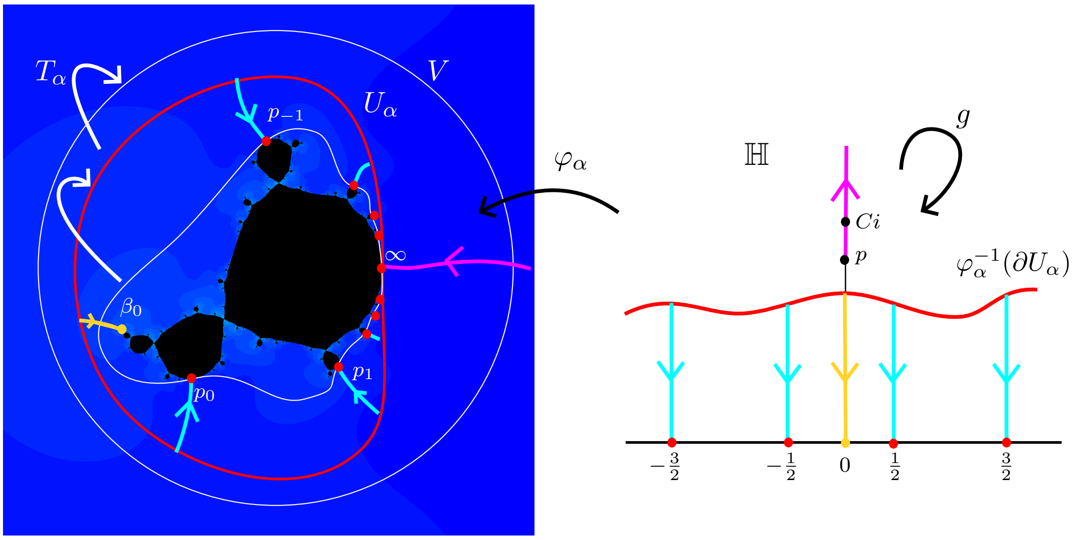

FIGURE 10. The conformal conjugacy on the basin of 0 (blue) and the labelling p_k of the poles (red dots) (1/2 corresponds to $k = 0$). The origin corresponds to the fixed point β_0.

Proof. Since ∂U_α is a Jordan curve and $V \setminus \overline{U_\alpha} \subset \Omega_\alpha$, it follows that ∞ is accessible from Ω_α for all α. Let $\gamma \in \Omega_\alpha$ be an arc landing at ∞. Then, by Lindeloff's Theorem (see e.g. [Pom92]) $\varphi_\alpha^{-1}(\gamma)$ is a curve in $\mathbb{H}$ landing at some point $w \in \partial\mathbb{H}$. Since ∞ is an essential singularity of T_α (and for example, all preimages of a given point accumulate at ∞), it follows that w is the only singularity of g in $\partial\mathbb{H}$, which is ∞. Again by Lindeloff's Theorem, the radial limit of φ_α at ∞ exists and equals ∞, hence (4.5) holds.

Let us now consider $T_\alpha^{-1}(\partial U_\alpha)$, which is a Jordan curve going through ∞ and containing all the poles of T_α. Since T_α is a covering from the pinched annulus $U_\alpha \setminus \overline{T_\alpha^{-1}(U_\alpha)}$ to $V \setminus \overline{U_\alpha}$, the curve γ has infinitely many disjoint preimages which are Jordan arcs joining ∂U_α and $\partial T_\alpha^{-1}(U_\alpha)$, ending at each pole of T_α. Again by Lindeloff's Theorem, the image of these segments under φ_α^{-1} must land at the distinct poles of g (since φ_α is a conjugacy), at which the radial limits exist and hence (4.6) holds.

To define β_0 observe that $w = 0$ is a repelling fixed point of g. By [BFJK17, Thm. A], the radial limit of φ_α at $w = 0$ exists and equals either ∞ or a weakly repelling fixed point of T_α. To rule out infinity, suppose that $\lim_{t\to 0} \varphi_\alpha(it) = \infty$. Then, since $g(it) = Ci\tanh(\pi t) \to 0$ as $t \to 0$, and φ_α is a conjugacy, it follows that $T_\alpha(\varphi_\alpha(it)) \to \infty$ as $t \to 0$, which implies that ∞ is an asymptotic value for T_α, a contradiction, since that occurs only for $\alpha = 0$. Hence β_0 is well defined as the landing point of the invariant arc $\varphi_\alpha(it)$ as $t \to 0$.

$\square$

4.3. **Rigidity of conjugacies near the Julia set.** The main result in this Section is Proposition 4.11, which shows that every quasiconformal conjugacy between T_α and itself defined near the filled Julia set, which fixes the prefered pole, extends to $K(f)$ as the identity. These result will only be used in Section 6.23.

We shall use notation and definitions from Section 4.2. We start by proving that topological conjugacies of g to itself near the real line extend to the real line as the identity, as long as they are close to the identity near one pole.

Lemma 4.8 (Real maps commuting with the tangent)**.** *Suppose that $h : \mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ is an increasing homeomorphism such that $h\circ g = g\circ h$ on $\mathbb{R}$ and $|h(\frac{1}{2})-\frac{1}{2}| < \frac{1}{2}$. Then $h = \mathrm{Id}$.*

Proof. First, observe that since $g(x+1) = g(x)$ for all $x \in \mathbb{R}$, we have $g \circ h(x+1) = g \circ h(x)$ for all $x \in \mathbb{R}$. But we also have that $g(x_1) = g(x_2)$ if and only if $x_1 - x_2 \in \mathbb{Z}$. Therefore, for every $x \in \mathbb{R}$, there exists $k_x \in \mathbb{Z}$ such that $h(x+1) - h(x) = k_x$. Since k_x is continuous and takes values in a discrete set, it is constant, hence there exists $k \in \mathbb{Z}$ such that for all $x \in \mathbb{R}$,

$$h(x+1) = h(x) + k. \tag{4.7}$$

On the other hand, $h(\infty) = \infty$, and h must map poles of g to other poles; so $h : \frac{1}{2}+\mathbb{Z} \to \frac{1}{2}+\mathbb{Z}$ is a monotone bijection. This implies that $|k| = 1$, and since h is increasing, $k = 1$. In particular, there exists $n_0 \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$, we have $h(\frac{1}{2}+n) = \frac{1}{2}+n_0+n$. But $|h(\frac{1}{2})-\frac{1}{2}| < \frac{1}{2}$ by hypothesis and hence $n_0 = 0$, i.e., all poles are fixed by h, and every interval $I_n := [-1/2+n, 1/2+n]$ is mapped bijectively to itself.

Now for any $j \in \mathbb{Z}$, let $k_j : \mathbb{R} \to I_j$ be the corresponding inverse branch of g^{-1} (explicitly, $k_j(z) = \frac{1}{\pi}\arctan(\frac{y}{C}) + j$). By the above, $h(I_j) = I_j$. Therefore, for any $j \in \mathbb{Z}$ and for any $x \in I_j$,

$$h(x) = k_j \circ h \circ g(x).$$

Let $x_j := \frac{1}{2} + j$ and $x_{j_2,j_1} := k_{j_2}(x_{j_1}) \in I_{j_2}$. Then

$$h(x_{j_2,j_1}) = k_{j_2} \circ h(x_{j_1}) = k_{j_2}(x_{j_1}) = x_{j_2,j_1}.$$

Proceeding inductively, it follows that if we let $x_{j_n,\dots,j_1} := k_{j_n} \circ \dots \circ k_{j_2}(x_{j_1})$, then $h(x_{j_n,\dots,j_1}) = x_{j_n,\dots,j_1}$ for all $n \in \mathbb{N}^*$ and all $(j_1,\dots,j_n) \in \mathbb{Z}^n$.

Since these points are dense in $\mathbb{R}$, we deduce $h = \mathrm{Id}$. ☐

Lemma 4.9 (Maps commuting in a neighborhood of $\mathbb{R}$)**.** *Let W be some strip in $\mathbb{H}$ with one boundary component being $\mathbb{R}$, such that $g^{-1}(W) \subset W$. Let $h : W \to W$ be quasiconformal homeomorphism such that $g \circ h = h \circ g$ on $g^{-1}(W)$. Let U be an open set in the upper half-plane $\mathbb{H}$ such that $\frac{1}{2} \in \partial U$, and suppose that $|h(z) - z| < 1/4$ for all $z \in U$. Then h extends continuously to $\mathbb{R}$ as the identity.*

Notice that the condition of $g^{-1}(W) \subset W$ ensures that the conjugacy makes sense.

Proof. Since h is a quasiconformal homeomorphism, it extends as a homeomorphism to the boundary of W (see e.g. Proposition 2.17 in [BF14]). We still denote by h the map $h : \overline{W} \to \overline{W}$ obtained via this extension. By continuity, we have that $h\circ g = g\circ h$ on $\mathbb{R}$ and, by hypothesis, $|h(z) - z| < 1/4$ for all $z \in U$. So h restricted to $\mathbb{R}$ satisfies the hypothesis of Lemma 4.8 and hence $h = \mathrm{Id}$ on $\mathbb{R}$. ☐

We are now ready to transport these theorems to the dynamical plane of T_α via the conformal conjugacies φ_α. We will make use of the following theorem by Krzyz [Krz68] (see [SV01, Theorem 2.19], for the special case of the unit disk).

Proposition 4.10. *Let $h : \mathbb{D} \to \mathbb{D}$ be a K-quasiconformal map which extends as the identity to $\partial\mathbb{D}$. Then for any $z \in \mathbb{D}$,*

$$\operatorname{dist}_{\mathbb{D}}(h(z), z) \le C, \tag{4.8}$$

Where C is a constant depending only on K.

Proposition 4.11 (Extension of conjugacies to $K(T_\alpha)$)**.** *Let $\alpha \in \mathbb{D}(0, 1/2)$ such that $K(T_\alpha)$ is connected. Let U, V be such that $T_\alpha : U \to V$ is TL, and let $p_0 = p_0(\alpha)$ the marked pole defined in (4.6) (Lemma 4.7). Let $h : V \setminus K(T_\alpha) \to V \setminus K(T_\alpha)$ be a quasiconformal homeomorphism such that*

$$h \circ T_\alpha = T_\alpha \circ h \qquad \text{on } U \setminus K(T_\alpha). \tag{4.9}$$

Then, there exists $\epsilon > 0$ such that: if there exists a domain $W' \subset U$ with $p_0 \in \partial W'$ and

$$\sup_{z \in W'} |h(z) - z| < \epsilon,$$

then h extends to a continuous homeomorphism $h : V \to V$ such that $h = \mathrm{Id}$ on $K(T_\alpha)$.

Proof. The homeomorphism h extends continuously to ∂V and hence it conmutes with T_α also on $\partial U_\alpha \setminus \{\infty\}$, where it is well defined. Hence, by setting $h(\infty) = \infty$, h is a homeomorphism of the closed anulus $A := \overline{V} \setminus U$, to itself, which conmutes with T_α. Let $\gamma(t) = \varphi_\alpha(i\,t)$ for $t > t_0$ large, where φ_α is the conformal conjugacy from Proposition 4.6. Then, by Lemma 4.7, $\gamma(t) \to \infty$ as $t \to \infty$. Since h is continuous on the closed annulus, $h(\gamma(t)) \to \infty$ as $t \to +\infty$. Let $\sigma_0(t)$ be the component of $T_\alpha^{-1}(\gamma(t))$ landing at p_0, which equals $\varphi_\alpha(\frac{1}{2} + it)$ for $t > 0$ small (see Lemma 4.7). Since h conmutes with T_α, we have that $h(\sigma_0(t))$ must be a curve whose image under T_α is $h(\gamma(t))$, hence it is landing at a pole.

Let $\epsilon > 0$ be such that the disc $D := \mathbb{D}(p_0, 2\epsilon)$ contains no other pole of T_α. Then $h(\mathbb{D}(p_0, \epsilon) \cap W') \subset D$ and hence $h(\sigma_0(t))$ lands at p_0.

Let $W := \varphi_\alpha^{-1}(V \setminus K(T_\alpha))$ which is a horizontal strip in $\mathbb{H}$ having $\mathbb{R}$ as its lower boundary component. Since φ_α conjugates g to T_α, it follows that $g^{-1}(W) = \varphi_\alpha^{-1}(U \setminus K_\alpha) \subset W$ and that the quasiconformal homeomorphism $\widetilde{h} = \varphi_\alpha^{-1} \circ h \circ \varphi_\alpha : W \to W$ commutes with g on $g^{-1}(W)$.

Since $\widetilde{h}$ is quasiconformal, it extends continuously to $\mathbb{R}$. We claim that $\widetilde{h}(\frac{1}{2}) = \frac{1}{2}$ and hence, by Lemma 4.9(or 4.8) $\widetilde{h}$ extends continuously to $\mathbb{R} \cup \{\infty\}$ by the identity. Indeed, both curves $\sigma_0(t)$ and $h(\sigma_0(t))$ are homotopic in Ω_α (because they belong to the pinched annulus $U \setminus \overline{T_\alpha^{-1}(U)}$ which is disjoint from the Julia set) and hence belong to the same access to p_0. Hence φ_α^{-1} sends them both to curves landing at the same point of $\mathbb{R} \cup \{\infty\}$ which needs to be $1/2$ by definition of σ_0. Since $\widetilde{h}$ commutes with g, the claim follows.

We now need to show that h extends to $J(T_\alpha)$ as the identity. Let us first show that there exists a constant $L > 0$ such that

$$\operatorname{dist}_W(\widetilde{h}(z), z) \le L \quad \text{for all } z \in W \text{ with } \operatorname{Im} z \text{ sufficiently small.} \tag{4.10}$$

Since $\widetilde{h}$ is the identity on $\mathbb{R} \cup \{\infty\}$, by mapping $\mathbb{H}$ conformally to $\mathbb{D}$ we obtain a map $h_{\mathbb{D}}$ which is defined on an annulus whose outer boundary is $\partial\mathbb{D}$, which is quasiconformal and which extends as the identity to $\partial\mathbb{D}$. Up to restricting V if necessary we can assume that ∂V is real analytic and hence ∂W is a quasicircle. Hence we can extend $h_{\mathbb{D}}$ as a quasiconformal map to all of $\mathbb{D}$, to which Proposition 4.10 applies. This gives that $\mathrm{dist}_{\mathbb{H}}(\widetilde{h}(z), z) \leq C$ for $z \in W$. Since the hyperbolic density in $\mathbb{H}$ and in W are comparable for z with small imaginary part, (4.10) follows.

Now, since $\varphi_\alpha : W \to V \setminus K(T_\alpha)$ is an isometry for the hyperbolic metric, (4.10) implies that

$$\mathrm{dist}_{V\setminus K(T_\alpha)}(h(z), z) \leq L \quad \text{for all } z \in V \setminus K(T_\alpha) \text{ close to } K(T_\alpha) \tag{4.11}$$

which implies that $|h(z) - z| \to 0$ as $z \to J(T_\alpha)$. Thus $h \to \mathrm{Id}$ as $z \to J(T_\alpha)$. $\square$

4.4. **Parameter plane: The Tandelbrot set. Proof of Theorem B.** As we saw in Lemma 4.4, for $\alpha \in \mathbb{D}(0, \frac{1}{2})$ the asymptotic value 1 belongs Ω_α, the basin of attraction of $z = 0$, and hence $1/\alpha$ acts as a free asymptotic value, whose orbit may or may not be captured by the basin of 0. In analogy to the Mandelbrot set, we define the following subset of the $\alpha-$parameter plane (see Figure 3).

Definition 4.12 (The Tandelbrot set)**.** The Tandelbrot set is

$$\mathcal{T} := \{\alpha \in \mathbb{C} \setminus \{1\} : \ T_\alpha^n\left(\frac{1}{\alpha}\right) \not\to 0 \text{ as } n \to \infty.\}.$$

As expected, the Tandelbrot set lies in a compact subset of the plane.

Lemma 4.13 ($\mathcal{T}$ is bounded)**.** $\mathcal{T} \subset \mathbb{D}(0, \frac{1}{2})$.

Proof. We need to show that if $|\alpha| \geq 1/2$ then $1/\alpha \in \Omega_\alpha$. On the one hand, if $1/2 \leq |\alpha| \leq 1$, $\alpha \neq 1$, then $1/\alpha \in \overline{\mathbb{D}}(0, 2)$ which, by Lemma 4.1, belongs to Ω_α. On the other hand, if $\alpha > 1$, consider the disk $D := \overline{\mathbb{D}}(0, |1/\alpha|)$ and argue as in the proof of Lemma 4.1 to show that $T_\alpha(D) \Subset D$, and hence $D \subset \Omega_\alpha$. $\square$

We shall see next that the boundary of $\mathcal{T}$ coincides with the activity locus of the asymptotic value $1/\alpha$. Note first that T_α is a natural family of meromorphic maps, with base map e.g. $T_0(z) = 1 - e^{-z/8}$, and holomorphic motions $\psi_\alpha(z) = z/(1-\alpha)$ and $\varphi_\alpha(z) = \frac{z}{\alpha z + 1 - \alpha}$. We recall the definitions of activity and passivity of singular values in Section 2 and define the *activity locus* of $1/\alpha$ as the set

$$\mathcal{A} = \{\alpha \in \mathbb{C} \setminus \{1\} \mid 1/\alpha \text{ is active at the parameter } \alpha\}.$$

Proposition 4.14 (Activity locus of $1/\alpha$)**.** *The boundary of $\mathcal{T}$ is the set of parameter values for which $1/\alpha$ is active. In other words*

$$\partial\mathcal{T} = \mathcal{A}.$$

Proof. Clearly $\partial\mathcal{T} \subset \mathcal{A}$. To see the reverse inclusion suppose that $\alpha_0 \in \mathcal{A}$ and let us show that there are parameters α arbitrarily close to α_0 such that $T_\alpha^n(1/\alpha) \to 0$ as $n \to \infty$. Since parameters in the activity locus cannot have a neighborhood on which $T_\alpha^n(1/\alpha) \to 0$, it will follow that α_0 is indeed in the boundary of $\mathcal{T}$.

To that end, observe that T_α is a natural family of meromorphic maps of finite type, which satisfies that $T_\alpha = \varphi_\alpha \circ T \circ \psi_\alpha^{-1}$, with $T(z) = \exp(z/8)$, $\psi_\alpha(z) = z/(\alpha - 1)$, and $\varphi(z) = (z-1)/(\alpha z - 1)$. By Proposition 2.20, every active parameter can be approximated by *virtual cycle parameters*, that is, by parameters α such that $T_\alpha^n(1/\alpha) = \infty$, for some $n \geq 1$ called the *order* of the virtual cycle.

At the same time, by the shooting lemma 2.22, every such virtual cycle parameter of order n can be approximated by parameters α such that $T_\alpha^{n+1}(1/\alpha) = 1/2$ (or any other constant function different from 1). Since $1/2$ is always in the basin of 0 for α in a neighborhood of $\mathcal{T}$, we get the result. □

Remark 4.15. By Lemma 4.2, the parameter space of the family $(T_\alpha)_{\alpha \in \mathbb{C}\setminus\{1\}}$ is invariant under the involution $\alpha \mapsto \frac{1}{\alpha}$ (except for $\alpha = 0$). Thus, the set $1/\mathcal{T}$ coincides with the set of $\alpha \in \mathbb{C} \setminus \{1\}$ such that the asymptotic value 1 is not captured by 0 for T_α. The full bifurcation locus is the union $\mathcal{T} \sqcup \partial(1/\mathcal{T})$. For this reason, we will only consider $\alpha \in \mathbb{D}$.

As for the Mandelbrot set, the interior of $\mathcal{T}$ consists of hyperbolic and (perhaps) non hyperbolic components; in the latter case, the Julia set supports an invariant line field.

Theorem 4.16 (Interior components of $\mathcal{T}$)**.** *Let U be a connected component of* $\mathrm{int}(\mathcal{T})$. *Then, one of the two following statements hold.*

(1) *For all $\alpha \in U$, T_α has an attracting cycle different from $z = 0$, or*
(2) *for all $\alpha \in U$, T_α has an invariant line field.*

In the first case, we say that U is a hyperbolic component; in that case, the multiplier map $\rho : U \to \mathbb{D}^*$ is a universal covering map. In the second case, we say that U is a non-hyperbolic component.

The claim about hyperbolic components follows from the results in [FK21] (see also [CJK23, Theorem 4.4]). For the second case we shall need the following lemma.

Lemma 4.17 (Holomorphic motion of J)**.** *Let U be a connected component of $\mathring{\mathcal{T}}$. Let $\alpha_* \in U$. Then the dynamical holomorphic motion of $J(T_{\alpha_*})$ over U extends to a holomorphic motion $h_\alpha : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is holomorphic on $\Omega(T_{\alpha_*})$.*

Proof. Let C, p, g, φ_α be as in Lemma 4.6. Recall that C, p, g do not depend on α and that $\varphi_\alpha : \mathbb{H} \to \Omega_\alpha$ is the Riemann map normalized so that $\varphi_\alpha(Ci) = 1$ and $\varphi_\alpha(p) = 0$, conjugating T_α in Ω_α to g on $\mathbb{H}$.

By J-stability the boundary of the family of punctured disks $(\Omega(T_\alpha), 0)$ moves holomorphically over U. By [Zak16, Main Theorem], it is enough to prove that the logarithm of conformal radius of $\Omega(T_\alpha)$ with respect to 0 is harmonic on U, i.e. that there exists a family of Riemann uniformizing maps $\psi_\alpha : \mathbb{D} \to \Omega(T_\alpha)$ such that $\psi_\alpha(0) = 0$ and $\alpha \mapsto \psi'_\alpha(0)$ is holomorphic on U, or equivalently, uniformizin maps $\widetilde{\psi}_\alpha : \mathbb{H} \to \Omega_\alpha$ such that $\widetilde{\psi}_\alpha(p) = 0$ and that $\alpha \mapsto \widetilde{\psi}'_\alpha(p)$ is holomorphic on U.

Let $\ell := \lim_{n\to\infty} 8^n g^n$ and $\ell_\alpha := \lim_{n\to\infty} 8^n T_\alpha^n$ denote the respective linearizing coordinates of g and T_α. Notice that $\ell'(p) = \ell'_\alpha(0) = 1$. Note also that

$$\ell_\alpha : \Omega_\alpha \longrightarrow \mathbb{C} \setminus (8^n a_\alpha)_{n\geq 1}$$

is a universal covering, where $a_\alpha := \ell_\alpha(1)$ depends holomorphically in α, since ℓ_α does. Likewise

$$\ell : \mathbb{H} \longrightarrow \mathbb{C} \setminus (8^n a)_{n\geq 1}$$

is also a universal covering, where $a := \ell(Ci)$ is independent of α. Finally define $L_\alpha(z) = \frac{a_\alpha}{a} z$ a linear map which sends the sequence $8^n a$ to $8^n a_\alpha$ for all n, and conjugates $z \mapsto z/8$ to itself.

Let $\widetilde{\psi}_\alpha : \mathbb{H} \to \Omega_\alpha$ be the lift of L_α under the universal coverings ℓ and ℓ_α, such that $\widetilde{\psi}_\alpha(p) = 0$, that is

$$\ell_\alpha \circ \widetilde{\psi}_\alpha = L_\alpha \circ \ell.$$

Since L_α is a homeomorphism, so is $\widetilde{\psi}_\alpha$. Using the chain rule (and recalling $\widetilde{\psi}(p) = 0$) we compute $\widetilde{\psi}'_\alpha(p) = \frac{L'_\alpha(\ell(0))\ell'(p)}{\ell'_\alpha(0)} = \frac{a_\alpha}{a}$. It follows that $\widetilde{\psi}_\alpha$ are Riemann maps sending p to 0 and such that $\widetilde{\psi}'_\alpha(p) = a_\alpha/a$, which depends holomorphically on α as required. (In fact, $\widetilde{\psi}_\alpha = \varphi_\alpha^{-1}$.) □

Proof of Theorem 4.16. Let $\alpha_0 \in U$. By [ABF21, Corollary G], if the orbit of $1/\alpha_0$ converges to an attracting cycle, the same holds for every parameter $\alpha \in U$. In that case, the period of the attracting cycle is constant throughout U and, by [FK21, Corollary 6.5], the multiplier map $\rho : U \to \mathbb{D}^*$ is a holomorphic covering. Moreover, U is either simply connected and ρ has therefore infinite degree, or U is conformally equaivalent to $\mathbb{D}^*$ and $\lim_{|w|\to 0} \rho^{-1}(w) = \alpha^*$, where $\alpha^* = 0$, the only parameter singularity in $\mathbb{D}(0, 1/2)$. We claim that the latter case is not possible, which finishes the proof of (a). Indeed, elementary calculations show that if $\alpha \in (0, \infty)$, then T_α is continuous, strictly increasing and concave in $\mathbb{R}^+$, a half line which is invariant under T_α. This implies that $T_\alpha(x) < x$ if $\alpha, x > 0$, and therefore $1/\alpha \in [0, \infty) \subset \Omega_\alpha$, the basin of 0, as we wanted to show.

Assume now that the orbit of $1/\alpha_0$ does not converge to an attracting cycle. The maps T_α are of finite type hence have no wandering domains. By the classification of periodic Fatou components and the fact that there are no Siegel disks or parabolic domains and no Herman rings (since the maps T_α have at most one free singular value), we must have ${}^c\Omega(T_\alpha) = J(T_\alpha)$ for all $\alpha \in U$. By [ABF21, Theorem E], there is a holomorphic motion h_α of the Julia set of T_{α_0}, respecting the dynamics. By Lemma 4.17, this holomorphic motion extends to a holomorphic motion h_α of the Riemann sphere which is holomorphic on $\Omega(T_{\alpha_*})$. Let μ_α denote the Beltrami coefficient of h_α. Then μ_α is supported in the Julia set. It cannot vanish a.e., for otherwise we would have that all maps in U would be linearly conjugated, contradicting Lemma 4.2. Therefore, μ_α is an invariant line field. □

Hence, analogously to the Mandelbrot set, the interior of the Tandelbrot is organized in hyperbolic components (also known as *shell components*) of constant period, and (conjecturaly non-existent) non-hyperbolic or non-hyperbolic components. The "main" component" visible in Figure 3, corresponds to parameters for which T_α has an attracting fixed point, and contains $\alpha = 0$ on its boundary, as we show in the following lemma. (See also [CJK22, Main Theorem] for a more general statement).

Lemma 4.18 (The main shell component). *If $\alpha < 0$, and $|\alpha|$ is small enough, T_α has a nonzero attracting fixed point. Hence the main hyperbolic component of $\mathcal{T}$ contains an interval $(\alpha_0, 0)$ for some $\alpha_0 < 0$.*

Proof. For any given $x_0 \in \mathbb{R}$ we have that $T_\alpha(x_0) \longrightarrow 1 - e^{-x_0/8}$ as $\alpha \to 0$. Let x^* be the unique nonzero fixed point of the map $h(x) := 1 - e^{-x/8}$, so that $h(x) < x$ for all $x < x^*$.

Choose $x_1 < x^*$ and let $\epsilon > 0$ be small enough so that for all $\alpha \in (-\epsilon, 0)$ we have $T_\alpha(x_1) < x_1$. Notice that $T_\alpha(x_1) < 1/\alpha$ since it is easy to check that T_α is strictly increasing and $T_\alpha(x) \to 1/\alpha$ as $x \to -\infty$.

Fix $\alpha \in (-\epsilon, 0)$ and choose $x_2 < 1/\alpha$. Then $T_\alpha(x_2) > 1/\alpha$ and thus $x_2 < T_\alpha(x_2) < T_\alpha(x_1) < x_1$. It follows that T_α has one fixed point $p = p(\alpha) \in (x_2, x_1)$ such that $T'_\alpha(p) \in (0, 1]$, since the graph must cross the diagonal at least one from above to below. However, if $T'_\alpha(p) = 1$ then $T''(p) = 0$ and it is easy to check that no inflection point can be a fixed point of T_α unless $\alpha = -1$. Hence $p(\alpha)$ is attracting and we have shown that $(\alpha_0, 0)$ is contained in a hyperbolic component of period one, where α_0 is the value up to which we can continue the fixed point $p(\alpha)$ before it becomes neutral. ☐

The following is the main Theorem in [CJK23].

Theorem 4.19. *The set $\mathcal{T}$ is connected and full.*

The approach in [CJK23] is to use quasiconformal surgery to prove that the complement of $\mathcal{T} \sqcup 1/\mathcal{T}$ is a topological annulus; the proof is relatively involved. We will give here a different proof, making use of an approximation of the maps T_α by rational maps and using the existing theory about unicritical polynomial-like maps.

As usual, let $V = \mathbb{D}(0, 2)$. Our goal is to show that the sets

$$A_n := \{\alpha \in \mathbb{D}(0, 1/2) \mid T^n_\alpha(1/\alpha) \in V\}$$

are connected and full. Then $\overline{A_n}$ is a decreasing sequence of compact, full, connected sets. Since

$$\mathcal{T} = \bigcap_{n\geq 0} \overline{A_n},$$

it follows that $\mathcal{T}$ is compact, connected and full.

We will study the properties of the sets A_n by approximating them with their rational analogues.

Definition 4.20 (A natural family of rational maps)**.** For $\alpha \in \mathbb{C} \setminus \{1\}$ and $k \in \mathbb{N}^*$, let $T_{\alpha,k} := M_\alpha \circ P_k \circ N_\alpha$, where $M_\alpha(z) := \frac{z-1}{\alpha z-1}$, $N_\alpha(z) := \frac{(\alpha-1)z}{8}$, and $P_k(z) := (1 + \frac{z}{k})^k$. Let $V := \{z \in \hat{\mathbb{C}} : |z| > 2\}$.

The following properties are straightforward to check:

Lemma 4.21 (Properites of $T_{\alpha,k}$)**.** *Let T_α, k be defined as above. Then,*

1. *$T_{\alpha,k}$ has two critical points of order $k-1$, ∞ and $c_{\alpha,k} := -\frac{8k}{\alpha-1}$; they are mapped to $\frac{1}{\alpha}$ and 1 respectively.*
2. *0 is a fixed point for $T_{\alpha,k}$, with multiplier $\frac{1}{8}$.*
3. *$T_{\alpha,k}(z) \to T_\alpha(z)$ as $k \to +\infty$, uniformly on every compact of $(\mathbb{C} \setminus \{1\}) \times \mathbb{C}$.*

For small values of α, maps in this rational family, are polynomial-like maps with good properties.

Lemma 4.22 (Maps $T_{\alpha,k}$ are polynomial-like)**.** *There exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, the family $T_{\alpha,k} : T^{-1}_{\alpha,k}(V) \to V$, $\alpha \in \mathbb{D}(0, \frac{1}{2})$, is a proper equipped holomorphic family of unicritical polynomial-like maps of degree k (see Definitions 2.12 and 2.11). The unique critical point is ∞, and the unique critical value is $\frac{1}{\alpha}$.*

Proof. Let us prove that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all $\alpha \in \mathbb{D}(0, \frac{1}{2})$, $T_{\alpha,k} : T^{-1}_{\alpha,k}(V) \to V$, $\alpha \in \mathbb{D}(0, \frac{1}{2})$ is a PL map with just ∞ as a critical point.

We first claim that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all $\alpha \in \mathbb{D}(0, \frac{1}{2})$, $T_{\alpha_k}(\mathbb{D}(0, 2)) \Subset \mathbb{D}(0, r)$, for some $0 < r < 2$. Indeed, this is true for T_α, and we know that $T_{\alpha,k}$ converges *uniformly* in (α, z) to $T_\alpha(z)$ for $(\alpha, z) \in \mathbb{D}(0, \frac{1}{2}) \times \mathbb{D}(0, 2)$. This proves $T_{\alpha,k}^{-1}(V) \Subset V$. Moreover, since $T_{\alpha,k}(c_{\alpha,k}) = 1$ and $1 \notin V$, and since $\frac{1}{\alpha} \in V$, V contains exactly one critical value (which is $\frac{1}{\alpha}$) and $T_{\alpha,k}^{-1}(V)$ contains exactly one critical point (which is ∞). Finally, $T_{\alpha,k} : T_{\alpha,k}^{-1}(V) \to V$ is a proper map, with exactly one critical value, hence $T_{\alpha,k}^{-1}(V)$ is simply connected, and is a Jordan domain. All this proves that for all $\alpha \in \mathbb{D}(0, \frac{1}{2})$, $T_{\alpha,k} : T_{\alpha,k}^{-1}(V) \to V$, $\alpha \in \mathbb{D}(0, \frac{1}{2})$ is indeed a unicritical PL map, whose critical point is ∞ and critical value is $\frac{1}{\alpha}$.

Next, let us prove that it is proper. By Lemma By definition of V, the winding property is trivially satisfied. By Lemma 4.4, for any $r \in (\frac{1}{2}, 1)$ we can replace the parameter space $\mathbb{D}(0, \frac{1}{2})$ with $\mathbb{D}(0, r)$ and $V = \hat{\mathbb{C}} \setminus \mathbb{D}(0, 2)$ with $\hat{\mathbb{C}} \setminus \mathbb{D}(0, \frac{1}{r})$. Finally, it is clear from the definition of V that $\alpha \mapsto \frac{1}{\alpha}$ winds once around ∞ as α turns once along $\partial\mathbb{D}(0, \frac{1}{2})$.

It remains to prove that this family of unicritical polynomial-like maps is equipped. Let $k \geq k_0$, and let $U_{\alpha,k} := T_{\alpha,k}^{-1}(V)$. Let $z_0 \in \partial U_{0,k}$. We make the following observation: for all $\alpha \in \mathbb{D}(0, \frac{1}{2})$, for all $z \in \partial V$, $M_\alpha^{-1}(z) \neq \infty$.

In particular, the continuous map

$$\mathbb{D}\left(0, \frac{1}{2}\right) \ni \alpha \mapsto M_\alpha^{-1} \circ T_{0,k}(z_0)$$

avoids ∞, which is the only critical value of $T_{0,k}$. Therefore, by the lifting property for covering maps, there exists a unique lift γ_{z_0} by $T_{0,k}$ of

$$\alpha \mapsto M_\alpha^{-1} \circ T_{0,k}(z_0)$$

such that $\gamma_{z_0}(0) = z_0$.

We then let $h_\alpha(z_0) := N_\alpha(\gamma_{z_0}(\alpha))$. By construction, we have

$$T_{\alpha,k} \circ h_\alpha(z_0) = T_{0,k}(z_0)$$

and $h_0(z_0) = z_0$.

By the Implicit Function Theorem, $\alpha \mapsto h_\alpha(z_0)$ is holomorphic on $\mathbb{D}\left(0, \frac{1}{2}\right)$. By the uniqueness of the lifting property, the map $z \mapsto h_\alpha(z)$ is injective for all $\alpha \in \mathbb{D}\left(0, \frac{1}{2}\right)$. Thus, $h : \mathbb{D}\left(0, \frac{1}{2}\right) \times \partial U_{0,k} \to \hat{\mathbb{C}}$ is a holomorphic motion, and $h_\alpha(\partial U_{0,k}) = \partial U_{\alpha,k}$.

Finally, since $\partial U_{\alpha,k}$ and ∂V are disjoint for every $\alpha \in \mathbb{D}\left(0, \frac{1}{2}\right)$, we may extend h_α to ∂V by setting

$$\widetilde{h}_\alpha = \begin{cases} h_\alpha(z) & \text{if } z \in \partial U_{0,k} \\ z & \text{if } z \in \partial V. \end{cases}$$

Applying Slodkowski's λ-lemma to $\widetilde{h}_\alpha$, we obtain a holomorphic motion of $\hat{\mathbb{C}}$ over $\mathbb{D}(0, \frac{1}{2})$, which we still denote by $\widetilde{h}_\alpha$, and which maps $\overline{V} \setminus U_{0,k}$ to $\overline{V} \setminus U_{\alpha,k}$. Moreover, it is equivariant by construction. □

The rational analogues of the sets A_n are the following.

Definition 4.23 (Rational approximants of A_n)**.** For $n \geq 0$ and $k \geq 2$, we define

$$A_{n,k} := \{\alpha \in \mathbb{D}(0, \frac{1}{2}) : T_{\alpha,k}^n(\frac{1}{\alpha}) \in V\}.$$

By Lemma 2.13, for every $k \geq k_0$ and every $n \in \mathbb{N}$, $A_{n,k}$ is a Jordan domain.

Definition 4.24 (Subsets of A_n)**.** For $\delta > 0$, we define

$$A_n(\delta) := \{\alpha \in A_n : |T_\alpha^j(\frac{1}{\alpha})| < \frac{1}{\delta}, \quad \forall\, 0 \le j \le n\}.$$

Observe that $A_n(\delta) \subset A_n$ and clearly, we have $A_n = \bigcup_{\delta>0} A_n(\delta)$.

For any set $X \subset \mathbb{C}$, we denote by X_ϵ the set $\{z \in \mathbb{C} : d(z, X) \le \epsilon\}$.

Lemma 4.25 ($A_{n,k}$ nearly converges to A_n)**.** *Let $n \in \mathbb{N}$. Then,*

(1) For all $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$,

$$\overline{A_{n,k}} \subset \left(\overline{A_n}\right)_\epsilon$$

(2) For all $\delta > 0$, for all $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$,

$$\overline{A_n(\delta)} \subset \left(\overline{A_{n,k}}\right)_\epsilon.$$

Proof. Let us first prove the first item. Let $\epsilon > 0$. We will prove that there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$, the complement of $\left(\overline{A_n}\right)_\epsilon$ is contained in the complement of $\overline{A_{n,k}}$.

Observe that all $\alpha \in \mathbb{C} \setminus \left(\overline{A_n}\right)_\epsilon$ are a definite distance away from any virtual cycle parameter. By continuity of $(\alpha, z) \mapsto T_\alpha(z)$ away from $z = \infty$, there exists $\eta > 0$ such that for all $0 \le j \le n$, for all $\alpha \in \mathbb{C} \setminus \left(\overline{A_n}\right)_\epsilon$:

$$\left|T_\alpha^j\left(\frac{1}{\alpha}\right)\right| \le \frac{1}{\eta} \quad \text{and} \quad \left|T_\alpha^n\left(\frac{1}{\alpha}\right)\right| \le 2 - \eta.$$

By uniform convergence of $(\alpha, z) \mapsto T_{\alpha,k}(z)$ to $T_\alpha(z)$ on compact subsets, there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$, for all $\alpha \in \mathbb{C} \setminus \left(\overline{A_n}\right)_\epsilon$:

$$\left|T_{\alpha,k}^n\left(\frac{1}{\alpha}\right)\right| \le 2.$$

This exactly proves the first item.

Let us now prove the second item. Let $\epsilon > 0$ and $\delta > 0$.

Let $\eta > 0$ be small enough that if $|T_{\alpha,k}^n(\frac{1}{\alpha})| > 2 - \eta$, then $d(\alpha, \overline{A_{k,n}}) \le \epsilon$. Again, by definition of $A_n(\delta)$, the sequence of maps $\alpha \mapsto T_{\alpha,k}^n(\frac{1}{\alpha})$ converge uniformly on $\overline{A_n(\delta)}$ to $T_\alpha^n(\frac{1}{\alpha})$. Therefore, there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$, for all $\alpha \in \overline{A_n(\delta)}$:

$$\left|T_{\alpha,k}^n\left(\frac{1}{\alpha}\right)\right| > 2 - \eta,$$

hence $\alpha \in \left(\overline{A_{k,n}}\right)_\epsilon$, and we are done. □

We are finally ready to prove the main goal and hence conclude the proof of Theorem 4.19.

Proposition 4.26. *For any $n \in \mathbb{N}$, the set $\overline{A_n}$ is connected and full.*

Proof. Assume for a contradiction that there is $n \in \mathbb{N}$ such that $\overline{A_n}$ is disconnected. Then there exists F_1, F_2 two disjoint compact sets such that $\overline{A_n} = F_1 \sqcup F_2$. Let $\delta > 0$ be small enough that $A_n(\delta) \cap F_i \ne \varnothing$, $1 \le i \le 2$. Such a δ exists since $A_n \supset A_n(\delta) \to A_n$ as $\delta \to 0$. Let $\epsilon := \frac{1}{4} d(F_1, F_2)$.

By Lemma 4.25, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\overline{A_{n,k}} \subset \left[\overline{A_n}\right]_\epsilon$. Since $\overline{A_{n,k}}$ is connected, and by our choice of ϵ, we have either $\overline{A_{n,k}} \subset (F_1)_\epsilon$ or $\overline{A_{n,k}} \subset (F_2)_\epsilon$. Assume without loss of generality that $\overline{A_{n,k}} \subset (F_1)_\epsilon$: then $d(F_2, \overline{A_{k,n}}) \geq \epsilon$.

On the other hand, applying the second item of Lemma 4.25 with $\frac{\epsilon}{2}$: there exists $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$,

$$\overline{A_n(\delta)} \subset \left[\overline{A_{n,k}}\right]_{\epsilon/2}.$$

Let $k \geq \max(k_0, k_1)$. Since $A_n(\delta) \cap F_2 \neq \varnothing$, there exists $\alpha \in F_2$ such that $d(\alpha, \overline{A_{n,k}}) \leq \frac{\epsilon}{2}$, a contradiction.

It remains to prove that $\overline{A_n}$ is full. Since $A_0 = \mathbb{D}(0, 1/2)$, the claim is true for $n = 0$. Let n be the smallest natural number for which $\overline{A_n}$ is not full, and let W denote a bounded connected component of $\mathbb{C} \setminus \overline{A_n}$, which by assumption is contained in A_{n-1}. Then, $T_\alpha^{n-1}(1/\alpha) \in U_\alpha$ for all $\alpha \in W$, which implies that the map $g(\alpha) := T_\alpha^n(1/\alpha)$ is meromorphic in W and different from 0 in W, since $\overline{U_\alpha}$ does not contain the zeros of T_α.

By definition of A_n, $|g(\alpha| \geq 2$ for all $\alpha \in \partial W \subset A_n$. Hence by the Minimum Modulus Principle, the same is true for all $\alpha \in W$, which implies that $W \subset A_n$. □

To end this section, we deduce Theorem B from the results that were proven until now.

Proof of Theorem B. The assertion that $\mathcal{T}$ is connected and full is Theorem 4.19. It is proved in Lemma 4.13 that $\mathcal{T} \subset \mathbb{D}(0, \frac{1}{2})$, so in particular $\mathcal{T} \subset \mathbb{D}$. Finally, by Proposition 3.7, for any tangent-like map (f, U, V, u, v), $J(f)$ is connected if and only if $v \in K(f)$; by Lemma 4.4, for any $\alpha \in \mathbb{D}$, there exists domains U_α, V_α such that $(T_\alpha, U_\alpha, V_\alpha, \infty, \frac{1}{\alpha})$ is tangent-like and the Julia set of T_α as a meromorphic map coincides with the Julia set of T_α as a tangent-like map (by Proposition 4.5). It then follows from the definition of $\mathcal{T}$ that $\alpha \in \mathcal{T}$ if and only if $J(T_\alpha)$ is connected. □

4.5. **Conjugacy classes in $\mathcal{T}$.** In this section we shall see that the Tandelbrot set $\mathcal{T}$ contains a unique representative of each hybrid conjugacy class (Theorem 4.28), while quasiconformal classes are either open or constant, the latter being the case for parameters in the boundary of $\mathcal{T}$ (Proposititon 4.31). We also show that local quasiconformal conjugacies can be upgraded to global ones (Corollary 4.30).

Definition 4.27 (Hybrid equivalent tangent maps)**.** For $\alpha_i \in \mathbb{D}$, we say that T_{α_1} and T_{α_2} are hybrid equivalent if there exist tangent-like restrictions $(T_{\alpha_1}, U_1, V_1, \infty, v_1)$ and $(T_{\alpha_2}, U_2, V_2, \infty, v_2)$ which are hybrid equivalent. That is, if there exists a quasiconformal homeomorphism $\psi : V_1 \to V_2$ such that for all $z \in U_1$,

$$\psi \circ f_1(z) = f_2 \circ \varphi(z)$$

and $\overline{\partial}\psi = 0$ a.e. on $K(T_{\alpha_1})$.

Notice that necessarily $\psi(v_1) = v_2$, and $\psi(\infty) = \infty$.

Theorem 4.28 (Hybrid conjugacy classes are singletons)**.** *Let T_{α_1} and T_{α_2} be hybrid equivalent, with $\alpha_i \in \mathcal{T}$. Then $\alpha_1 = \alpha_2$.*

To prove this Theorem we shall construct an affine conjugacy between T_{α_1} and T_{α_2}, as a limit of quasiconformal maps. To start the construction, let U_i, V_i such that $T_{\alpha_i} : U_i \to V_i$ are tangent-like (see Lemma 4.4) and let $\psi : V_1 \to V_2$ denote a hybrid conjugacy. Notice that

ψ fixes infinity. Let $\varphi : \Omega_1 \to \Omega_2$ denote the conformal conjugacy given by lemma 4.6 where Ω_i denote the attracting basins of 0 for T_{α_i}. Observe that there is no reason for ∂V_1 to be mapped to ∂V_2 under φ, but this poses no problem in the construction that follows.

Consider a quasiconformal interpolation (see Lemma 2.5) h between $\varphi|_{\partial V_1}$ and $\psi|_{\partial U_1}$ on the annulus $V_1 \setminus \overline{U}_1$ and define the quasiconformal homeomorphism $h_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as

$$h_0 = \begin{cases} \psi & \text{on } \overline{U}_1 \\ h & \text{on } V_1 \setminus \overline{U}_1 \\ \varphi & \text{on } \hat{\mathbb{C}} \setminus V_1. \end{cases}$$

Note that h_0 is not a conjugacy on all of $\hat{\mathbb{C}}$, but it is a quasiconformal conjugacy on $U_1 \supset K(T_{\alpha_1})$ satisfying $\overline{\partial}\psi = 0$ on $K(T_{\alpha_1})$, and a holomorphic conjugacy on $\hat{\mathbb{C}} \setminus \overline{V_1}$.

The sequence of conjugacies is defined in the following lemma.

Lemma 4.29. *There exists a sequence of uniformly quasiconformal homeomorphisms h_i : $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that for all i*

$$h_i = \psi \qquad \text{on } T_{\alpha_1}^{-i}(\overline{U}_1), \tag{4.12}$$

$$h_i = \varphi \qquad \text{on } T_{\alpha_1}^{-i}(\hat{\mathbb{C}} \setminus V_1) \tag{4.13}$$

$$h_i \circ T_{\alpha_1} = T_{\alpha_2} \circ h_{i+1} \qquad \text{on } \hat{\mathbb{C}}. \tag{4.14}$$

Proof of Lemma 4.29. Let h_0 be defined as above and assume that h_i has been constructed for some $i \in \mathbb{N}$. Notice that $h_i(\frac{1}{\alpha_1}) = \frac{1}{\alpha_2}$ and that $h_i(1) = 1$. Indeed, $h_i = \varphi$ is a (holomorphic) conjugacy on an open subset of the basin of 0 containing 1; and $h_i = \psi$ is a conjugacy on $T_{\alpha_1}^{-i}(\overline{U}_1) \supset K(T_{\alpha_1})$, which contains $\frac{1}{\alpha_1}$ because the filled Julia set is connected (see Prop 3.7 and 4.5). Also $h_i(0) = 0$ by the induction hypothesis.

The maps $h_i \circ T_{\alpha_1} : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{1, \frac{1}{\alpha_2}\}$ and $T_{\alpha_2} : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{1, \frac{1}{\alpha_2}\}$ are then two universal covers. So (see e.g. [Hat02, Proposition 1.37]) there exists a unique homeomorphism $h_{i+1} : \mathbb{C} \to \mathbb{C}$ such that

$$h_i \circ T_{\alpha_1} = T_{\alpha_2} \circ h_{i+1},$$

normalized by $h_{i+1}(0) = 0$. It extends to a homeomorphism $h_{i+1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing ∞ which satisfies (4.14) by construction, hence is quasiconformal with the same dilatation as h_i. It remains to show that $h_{i+1} = h_i = \psi$ on $T_{\alpha_1}^{-(i+1)}(\overline{U}_1)$ and that $h_{i+1} = h_i = \varphi$ on $T_{\alpha_1}^{-(i+1)}(\hat{\mathbb{C}} \setminus V_1)$.

By the induction hypothesis $h_i = \psi$ on $T_{\alpha_1}^{-i}(\overline{U}_1)$, hence $T_{\alpha_2} \circ h_i = h_i \circ T_{\alpha_1}$ on $T_{\alpha_1}^{-(i+1)}(\overline{U}_1) \subset T_{\alpha_1}^{-i}(\overline{U}_1)$. Moreover, both

$$T_{\alpha_2} : T_{\alpha_2}^{-(i+1)}(\overline{U}_2) \to T_{\alpha_2}^{-i}(\overline{U}_2) \setminus \{1/\alpha_2\}$$

and

$$h_i \circ T_{\alpha_1} : T_{\alpha_1}^{-(i+1)}(\overline{U}_1) \to T_{\alpha_2}^{-i}(\overline{U}_2) \setminus \{1/\alpha_2\}$$

are also universal covers. If we fix some some arbitrary $z_0 \in T_{\alpha_1}^{-(i+1)}(\overline{U_1})$, then by the induction hypothesis, $h_i \circ T_{\alpha_1}(z_0) = \psi \circ T_{\alpha_1}(z_0)$, and since ψ is a conjugacy,

$$h_i \circ T_{\alpha_1}(z_0) = T_{\alpha_2} \circ \psi(z_0).$$

By the unicity of the equivalence of covering maps ([Mun18, Theorem 79.2 p. 480]), we therefore have $h_{i+1} = \psi$ on $T_{\alpha_1}^{-(i+1)}(\overline{U_1})$.

Likewise, if we define the Jordan domain $\widetilde{V_2} := \hat{\mathbb{C}} \setminus \varphi(\hat{\mathbb{C}} \setminus V_1)$, then

$$T_{\alpha_2} : T_{\alpha_2}^{-(i+1)}(\hat{\mathbb{C}} \setminus \widetilde{V_2}) \to T_{\alpha_2}^{-i}(\hat{\mathbb{C}} \setminus \widetilde{V_2}) \setminus \{1\}$$

and

$$h_i \circ T_{\alpha_1} : T_{\alpha_1}^{-(i+1)}(\hat{\mathbb{C}} \setminus V_1) \to T_{\alpha_2}^{-i}(\hat{\mathbb{C}} \setminus V_1) \setminus \{1\}$$

are both universal covers. Again by unicity of the equivalence between universal covers, we must have $h_{i+1} = h_i = \varphi$ on $T_{\alpha_1}^{-(i+1)}(\hat{\mathbb{C}} \setminus V_1) \subset \Omega_1$. □

We are now ready for the proof.

Proof of Theorem 4.28. Let h_i be the sequence of quasiconformal homeomorphisms given by Lemma 4.29. By compactness of quasiconformal maps with uniformly bounded dilatation, we can extract a subsequence converging uniformly to a quasiconformal homeomorphism $h_\infty : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Moreover, $h_\infty = \varphi$ on Ω_1, and $h_\infty = \psi$ on $K(T_{\alpha_1})$. Therefore, $\overline{\partial} h_\infty = 0$ a.e., so by Weyl's lemma, φ_∞ is analytic on $\hat{\mathbb{C}}$, hence affine (since it fixes ∞). So T_{α_1} and T_{α_2} are affinely conjugated, which, by Lemma 4.2 implies that $\alpha_1 = \alpha_2$. □

A consequence of the proof is the following corollary.

Corollary 4.30 (From local to global quasiconformal conjugacy)**.** *Let $T_{\alpha_1}, T_{\alpha_2}$ be two tangent maps which are quasiconformally conjugate on V. Then they are quasiconformally conjugate on $\hat{\mathbb{C}}$.*

Proof. We proceed as in the proof of Theorem 4.28, and h_∞ provides the desired conjugacy, which in this case is not necessarily an affine map. □

This allows us to describe the quasiconformal conjugacy classes in the model family.

Proposition 4.31 (Quasiconformal conjugacy classes)**.** *Quasiconformal conjugacy classes in $\mathcal{T}$ are either open or constant. Moreover, if $\alpha \in \partial\mathcal{T}$, then α is the unique representative in its quasiconformal conjugacy class.*

Proof. Let $\alpha_i \in \mathcal{T}$, $i = 0, 1$, and T_{α_0} and T_{α_1} be quasiconformally conjugate on a neighborhood of the Julia set, by a quasiconformal map φ. By Corollary 4.30 we may assume that φ is a global conjugacy in all of $\hat{\mathbb{C}}$. Let μ be the Beltrami form induced by φ on the dynamical plane of T_{α_0}, and consider the holomorphic family of Beltrami forms $\{t\mu, t \in \mathbb{D}(0, ||\mu||_\infty^{-1})\}$. Let φ_t be the integrating maps of $t\mu$ fixing $0, 1$ and ∞. Then the maps φ_t depend holomorphically on t and conjugate T_{α_0} to a family of tangent maps $T_{\alpha(t)}$ such that $t \mapsto \alpha(t)$ is holomorphic, $\alpha(0) = \alpha_0$ and $\alpha(1) = \alpha_1$. Since holomorphic maps are open or constant, the first statement is proven.

Now observe that $J(T_\alpha)$ is connected if and only if $\alpha \in \mathcal{T}$. Therefore, if $\alpha \in \partial\mathcal{T}$ and its quasiconformal conjugacy class is open, $J(T_{\alpha'})$ is connected for α' in a neighborhood of α, a contradiction.

□

We will end this section with a technical lemma, Lemma 4.33, which shows that a conjugacy between T_α and itself in any fundamental annulus, extends uniquely to a conjugacy up to the filled Julia set. It will be used in Section 6.3.3 (Proposition 6.23) when dealing with the parameter version of the Straightening Theorem.

This result is similar to a corresponding lemma for quadratic-like maps with connected Julia sets, and However, our situation is more technical, due to the fact that if $T_\alpha : U \to V$ is TL with connected Julia set, then pullbacks of the fundamental annulus $\overline{V} \setminus U$ do not remain topological annuli, but annuli that are "pinched" at prepoles. See Figure 7 (left). Nevertheless the closure of these sets are still connected and contractible to a circle.

Lemma 4.32 (Pinched annuli)**.** *Let $D, D' \subset \mathbb{C}$ be Jordan domains such that $D' \subset D$. Then $X := \overline{D} \setminus D'$ is connected, and $\pi_1(X) \simeq \mathbb{Z}$.*

Observe that we do not assume that $D' \Subset D$: there can be many intersections between the boundaries of D and D'.

Proof. We will prove that X is a deformation retract of a closed annulus, which implies both its connectedness and the fact that $\pi_1(X) \simeq \mathbb{Z}$. Let $\varphi : \mathbb{D} \to D'$ be a Riemann map. Since by assumption D' is a Jordan domain, φ extends to a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{D'}$. For $0 < t \leq 1$, let $A_t := \overline{D} \setminus \varphi(\mathbb{D}(0,t))$. For any $0 < t < 1$, $D_t := \varphi(\mathbb{D}(0,t))$ is a Jordan domain compactly contained in D, therefore A_t is a closed annulus; and of course $X \subset A_t$.

For $0 \leq t \leq 1$, we define $F_t : A_{1/2} \to X$ by

$$F_t(x) = \begin{cases} x & \text{if } x \in X \\ \varphi\left((1-t)\varphi^{-1}(x) + t\frac{\varphi^{-1}(x)}{|\varphi^{-1}(x)|}\right) & \text{otherwise.} \end{cases}$$

Then:

1. $F_t(x) = x$ for all $x \in X$, by definition
2. For all $0 \leq t \leq 1$, $F_t(A_{1/2}) \subset A_{1/2}$
3. For all $x \in A_{1/2}$, $F_0(x) = x$
4. For all $x \in A_{1/2}$, $F_1(x) \in \partial D' \subset X$.

Therefore X is indeed a deformation retract of $A_{1/2}$. □

Lemma 4.33 (Extension of conjugacies on fundamental annuli)**.** *Let $\alpha \in \mathcal{T}$ be such that T_α has no virtual cycle. Let (U_1, V_1) and (U_2, V_2) be two pairs of Jordan domains such that $(T_\alpha, U_i, V_i, \infty, \frac{1}{\alpha})$ are two TL restrictions of T_α. Let $h : \overline{V_1} \setminus U_1 \to \overline{V_2} \setminus U_2$ be a homeomorphism such that for all $z \in \partial U_1 \setminus \{\infty\}$,*

$$h \circ T_\alpha(z) = T_\alpha \circ h(z).$$

Then h extends uniquely to a homeomorphism $\widetilde{h} : \overline{V_1} \setminus K(T_\alpha) \to \overline{V_2} \setminus K(T_\alpha)$ such that for all $z \in U_1 \setminus K(T_\alpha)$,

$$\widetilde{h} \circ T_\alpha(z) = T_\alpha \circ \widetilde{h}(z).$$

Proof. We start by introducing some notations: for $n \in \mathbb{N}$ and $i \in \{1, 2\}$, let $C_n^i := \overline{T_\alpha^{-n}(\partial V_i)}$, $D_n^i := T_\alpha^{-n}(V_i)$ and $X_n^i := \overline{D_n^i} \setminus D_{n+1}^i$.

We claim that the sets $D_{n,i}$ are Jordan domains. We will argue inductively on $n \in \mathbb{N}$. By definition of a TL map, D_0^i is a Jordan domain. Let now $n \in \mathbb{N}$ be such that D_n^i is a Jordan domain. Then, since $\frac{1}{\alpha} \in D_{n,i}$, the set $D_{n+1}^i := T_\alpha^{-1}(D_n^i)$ is a logarithmic tract above D_n^i, hence a Jordan domain.

We can now apply Lemma 4.32: the sets X_n^i are connected, and $\pi_1(X_n^i) \simeq \mathbb{Z}$. Moreover, the sets C_n^i are the boundary of D_n^i, so they are Jordan curves.

Next, we construct a sequence of homeomorphisms $h_n : X_n^1 \to X_n^2$ such that $h_0 = h$ and for all $n \geq 1$,

(1) for all $z \in X_n^1 \setminus \{\infty\}$, $h_n \circ T_\alpha(z) = T_\alpha \circ h_{n-1}(z)$
(2) $h_n = h_{n-1}$ on $X_n^1 \cap X_{n-1}^1$.

We first treat the case $n = 1$. The sets X_0^i are closed annuli which do not contain any singular value, so $T_\alpha : T_\alpha^{-1}(X_0^i) \to X_0^i$ is a covering map. It has infinite degree, so by the classification of coverings of annuli, it must be a universal cover. Then both maps $T_\alpha : T_\alpha^{-1}(X_0^2) \to X_0^2$ and $h_0 \circ T_\alpha : T_\alpha^{-1}(X_0^1) \to X_0^2$ are universal covers. Let $z_0 \in C_1^1 \setminus \{\infty\}$: then $z_0 \in X_0^1 \cap X_1^1$. By the unicity of universal covers and the unicity of the equivalence between universal covers, there is a unique homeomorphism $h_1 : T_\alpha^{-1}(X_0^1) \to T_\alpha^{-1}(X_0^2)$ such that $h_1(z_0) = h_0(z_0)$ and $h_1 \circ T_\alpha(z) = T_\alpha \circ h_0(z)$ for all $z \in T_\alpha^{-1}(X_0^1)$. We can extend continuously h_1 to $X_1^1 = T_\alpha^{-1}(X_0^1) \cup \{\infty\}$ be setting $h_1(\infty) := \infty$. This proves (1).

It remains to prove that h_1 coincides with h_0 on $X_1^1 \cap X_0^1 = C_1^1$. This follows from the same unicity argument: both $T_\alpha \circ h_0 : C_1^1 \setminus \{\infty\} \to C_0^1$ and $h_1 \circ T_\alpha : C_1^1 \setminus \{\infty\} \to C_0^1$ are universal covers, and they coincide at $z_0 \in C_1^1$ by construction; so they coincide on all of $C_1^1 \setminus \{\infty\}$. And since $h_1(\infty) = h_0(\infty) = \infty$, h_1 and h_0 coincide on all of $X_1^1 \cap X_0^1$. This proves (2).

We now treat the general case, which is exactly the same. Let $n \geq 1$ be such that h_n is constructed. As we have proved, the sets X_n^i are connected and $\pi_1(X_n^i) \simeq \mathbb{Z}$, and moreover X_n^i does not contain any asymptotic value. Therefore, $T_\alpha : T_\alpha^{-1}(X_n^i) \to X_n^i$ is a covering map. It has infinite degree, so by the classification of coverings, it must be a universal cover. Then both maps $T_\alpha : T_\alpha^{-1}(X_n^2) \to X_n^2$ and $h_n \circ T_\alpha : T_\alpha^{-1}(X_n^1) \to X_n^2$ are universal covers. Let $z_n \in C_{n+1}^1 \setminus \{\infty\}$: then $z_n \in X_n^1 \cap X_{n+1}^1$. By the unicity of universal covers and the unicity of the equivalence between universal covers, there is a unique homeomorphism $h_{n+1} : T_\alpha^{-1}(X_n^1) \to T_\alpha^{-1}(X_n^2)$ such that $h_{n+1}(z_n) = h_n(z_n)$ and $h_{n+1} \circ T_\alpha(z) = T_\alpha \circ h_n(z)$ for all $z \in T_\alpha^{-1}(X_n^1)$. We can extend continuously h_{n+1} to $X_{n+1}^1 = T_\alpha^{-1}(X_0^1) \cup \{\infty\}$ be setting $h_{n+1}(\infty) := \infty$. This proves (1).

It remains to prove that h_{n+1} coincides with h_n on $X_{n+1}^1 \cap X_n^1 = C_{n+1}^1$. This follows from the same unicity argument: both $T_\alpha \circ h_n : C_{n+1}^1 \setminus \{\infty\} \to C_n^1$ and $h_{n+1} \circ T_\alpha : C_{n+1}^1 \setminus \{\infty\} \to C_n^1$ are universal covers, and they coincide at $z_n \in C_{n+1}^1$ by construction; so they coincide on all of $C_{n+1}^1 \setminus \{\infty\}$. And since $h_{n+1}(\infty) = h_n(\infty) = \infty$, h_{n+1} and h_n coincide on all of $X_{n+1}^1 \cap X_n^1$. This proves (2).

Finally, we set $\widetilde{h}(z) := h_n(z)$ if $z \in X_n^1$. By condition (2), the map $\widetilde{h}$ is well-defined and continuous on $\bigcup_{n\geq 0} X_n^1 = V_1 \setminus K(T_\alpha)$. By construction, it is a homeomorphism $\widetilde{h} : V_1 \setminus K(T_\alpha) \to V_2 \setminus K(T_\alpha)$, and it satisfies

$$\widetilde{h} \circ T_\alpha(z) = T_\alpha \circ \widetilde{h}(z)$$

for all $z \in U_1 \setminus K(T_\alpha)$. □

5. The straightening theorem

The goal in this section is to prove Theorem A: The Straightening Theorem for tangent-like maps. We end the section with some considerations about the uniqueness of the straightening maps.

Theorem 5.1 (Straightening theorem)**.** *Let (f, U, V, u, v) be a tangent-like map. Then there exists $\alpha \in \mathbb{C} \setminus \{1\}$ such that f is hybrid equivalent to T_α. Moreover, if $K(f)$ is connected, then α is unique, and $\alpha = 0$ if and only if $u = v$.*

In the following proof, we take as model map the map $T_0(z) = 1 - e^{-\frac{z}{8}}$. We could in fact choose here any T_α, for $\alpha \in \mathcal{T}$. This map will play the role that is played by the external map z^d in the classical straightening theorem by Douady and Hubbard; for tangent-like maps, the role of the superattracting basin of infinity will be replaced by the role of the the attracting basin of 0.

Proof. Let (f, U, V, u, v) be a tangent-like map. Up to conjugating with a conformal map (which is a hybrid conjugacy) we can assume that V is a Euclidean disk containing infinity and that $u = \infty$. Indeed, since V is a simply connected domain with analytic boundary, we can map it conformally to a Euclidean disk containing infinity in such a way that u is mapped to infinity; the aforementioned map extends analytically to the boundary and the image of U is a quasicircle.

Let $T_0(z) = 1 - e^{-\frac{z}{8}}$. By Lemma 4.4, (T_0, U_0, V_0) is a tangent-like map for $V_0 = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}(0, 2)}$, $U_0 = T_0^{-1}(V_0)$, $u_0 = v_0 = \infty$. Let t_1 be a Möbius map which maps V to V_0 sending v to $v_0 = \infty$. Note that t_1 is globally defined and maps $\hat{\mathbb{C}} \setminus \overline{V}$ to $\hat{\mathbb{C}} \setminus \overline{V_0}$.

Since $f : \overline{U} \setminus \{\infty\} \to \overline{V} \setminus \{v\}$ and $T_0 : \overline{U_0} \setminus \{\infty\} \to \overline{V_0} \setminus \{v_0\}$ are universal covers, the homeomorphism $t_1 : \overline{V} \setminus \{v\} \to \overline{V_0} \setminus \{v_0\}$ lifts as a homeomorphism $t_2 : \overline{U} \setminus \{\infty\} \to \overline{U_0} \setminus \{\infty\}$ such that

$$T_0 \circ t_2 = t_1 \circ f.$$

The lift $t_2 : \partial U \setminus \{\infty\} \to \partial U_0 \setminus \{\infty\}$ is quasisimmetric since it is the boundary extension of a conformal map (see Lemma 2.4).

By the interpolation lemma 2.5, there exists a quasiconformal extension $t : \overline{V} \setminus U$, interpolating between $t_{1|\partial V}$ and $t_{2|\partial U}$. Extending as $t = t_1$ on the complement of V, we now have a quasiconformal map $t : \hat{\mathbb{C}} \setminus U \to \hat{\mathbb{C}} \setminus U_0$, which is a Moebius map on $\hat{\mathbb{C}} \setminus V$, and such that $T_0 \circ t = t \circ f$ on ∂U.

We shall use t to glue T_0 on the complement of U. To that end, define the quasiregular map $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as

$$F(z) = \begin{cases} f(z) & \text{if } z \in U \\ t^{-1} \circ T_0 \circ t(z) & \text{if } z \notin U, \end{cases}$$

Since F is conformally conjugate to T_0 via t on $\hat{\mathbb{C}} \setminus \overline{V}$, $z^* = t^{-1}(0)$ is an attracting fixed point for F of multiplier $1/8$.

The rest of the proof is the same as in the case of the classical straightening theorem. Let $\mu_0 \equiv 0$ denote the trivial Beltrami form and $A := V \setminus \overline{U}$, the only part of the plane where F is not holomorphic but quasiregular. We define an F-invariant Beltrami form as follows. Let $\mu := t^* \mu_0$ on $\hat{\mathbb{C}} \setminus U$ (hence $\mu = \mu_0$ on $\hat{\mathbb{C}} \setminus V$). Extend μ to $U \setminus K(f)$ by pull-backs under the map F i.e., abusing notation, $\mu = (F^n)^* \mu$ on $F^{-n}(A)$ and, finally, $\mu = 0$ on $K(f)$. Then μ has bounded dilatation and is $F-$invariant.

Let $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasiconformal map integrating μ (i.e. satisfying $\varphi^* \mu_0 = \mu$) provided by the Measurable Riemann Mapping Theorem, normalized so that $\varphi(\infty) = \infty$, $\varphi(z^*) = 0$ and $\varphi((t^{-1}(1)) = 1$. Then $G := \varphi \circ F \circ \varphi^{-1}$ is meromorphic transcendental, and quasiconformally conjugate to F under φ which, since $\mu = 0$ on $K(f)$, is actually a hybrid equivalence. Moreover,

G has exactly two singular values (1 and $\varphi(v)$), both of which are omitted, and one fixed point 0 with multiplier $1/8$, since φ is conformal outside V. By Lemma 4.2, G is affine conjugate to T_α for some α. In fact, by our choice of normalization, $G = T_\alpha$ with $\alpha = 1/\varphi(v)$.

Now suppose that $K(f)$ is connected and that there are α_1 and α_2 such that f is hybrid equivalent to both T_{α_1} and T_{α_2}. Then both maps are hybrid equivalent to each other and $\alpha_i \in \mathcal{T}$, $i = 1, 2$. Hence by Theorem 4.28, $\alpha_1 = \alpha_2$.

The last claim follows from the fact that $u = v$ if and only if the straightened map T_α has ∞ as an asymptotic value. Since the asymptotic values of T_α are 1 and $\frac{1}{\alpha}$, this is equivalent to $\alpha = 0$. $\square$

We record here the following fact, which will play an important role later on in the proof of Proposition 6.30,, when dealing with the parameter version of the Straightening Theorem. It is more a by-product of the proof of Theorem 5.1 than a consequence of its statement.

Corollary 5.2 (Towards Böttcher coordinates). *With the same notations as in the proof of Theorem 5.1: the map $\Psi_\alpha := t \circ \varphi^{-1}$ is holomorphic on $\hat{\mathbb{C}} \setminus \overline{\varphi(U)}$ and conjugates T_α to T_0. Moreover, Ψ_α depends only on α, in the sense that if $(\widetilde{f}, \widetilde{U}, \widetilde{V}, \widetilde{u}, \widetilde{v})$ is another tangent-like map hybrid equivalent to T_α and $\widetilde{\Psi}_\alpha := \widetilde{t} \circ \widetilde{\varphi}^{-1}$, then $\widetilde{\varphi}(\widetilde{U}) = \varphi(U)$ and $\widetilde{\Psi}_\alpha = \Psi_\alpha$ on $\hat{\mathbb{C}} \setminus \overline{\varphi(U)}$.*

Proof. Using the notation of the proof of the Straightening Theorem, define $\Psi_\alpha := \psi \circ \varphi^{-1} : \hat{\mathbb{C}} \setminus \overline{\varphi(U)} \to \hat{\mathbb{C}} \setminus \overline{U_0}$. By construction, the maps φ and t have the same Beltrami coefficient on the annulus $\overline{V} \setminus U$; moreover, φ and t are holomorphic on $\hat{\mathbb{C}} \setminus \overline{V}$. Therefore Ψ is holomorphic on $\hat{\mathbb{C}} \setminus \overline{\varphi(U)}$. The fact that Ψ_α conjugates T_α to T_0 is also by construction.

It remains to prove the uniqueness. Let $\Psi := (\widetilde{\Psi}_\alpha)^{-1} \circ \Psi_\alpha : \hat{\mathbb{C}} \setminus \overline{\varphi(U)} \to \hat{\mathbb{C}} \setminus \overline{\widetilde{\varphi}(\widetilde{U})}$. Then Ψ is biholomorphic, and commutes with T_α: for all $z \in \hat{\mathbb{C}} \setminus \overline{\varphi(U)}$,

$$\Psi \circ T_\alpha(z) = T_\alpha \circ \Psi(z).$$

Since $T_\alpha : \hat{\mathbb{C}} \setminus \overline{\varphi(U)} \to \left(\hat{\mathbb{C}} \setminus \overline{\varphi(V)}\right) \setminus \{1\}$ is a universal cover, and similarly $T_\alpha : \hat{\mathbb{C}} \setminus \overline{\widetilde{\varphi}(\widetilde{U})} \to \left(\hat{\mathbb{C}} \setminus \overline{\widetilde{\varphi}(\widetilde{V})}\right) \setminus \{1\}$ is also a universal cover, it follows that $\Psi(1) = 1$. Then, for all $n \in \mathbb{N}$, $\Psi \circ T_\alpha^n(1) = T_\alpha^n(1)$, and so by the identity principle, $\Psi = \mathrm{Id}$ and $\varphi(U) = \widetilde{\varphi}(\widetilde{U})$.

$\square$

We conclude this section showing that, although the straightening map of a TL map is not unique, the set of possible restrictions to the Julia set is somehow "discrete".

Proposition 5.3 (Uniqueness of (nearby) hybrid conjugacies). *Suppose ψ_1 is a hybrid equivalence between a TL map (f, U, V, ∞, v) and T_α for some $\alpha \in \mathcal{T}$. Let p denote the marked pole of T_α (see Lemma 4.7) and let $p_1 := \psi_1^{-1}(p)$. Then, there exists $\epsilon > 0$ such that for every domain $W \subset U$ such that $p_1 \in \partial W$, for every hybrid equivalence ψ_2 between f and T_α, if $|\psi_1(z) - \psi_2(z)| < \epsilon$ on W, then $\psi_1 = \psi_2$ on $J(f)$.*

Proof. Let ψ_2 be a hybrid conjugacy between f and T_α. Let $V_i := \psi_i(V)$ and $U_i := \psi_i(U)$, $1 \leq i \leq 2$. The map $\psi := \psi_2 \circ \psi_1^{-1} : V_1 \to V_2$ is quasiconformal, maps U_1 to U_2 and satisfies

$$T_\alpha \circ \psi(z) = \psi \circ T_\alpha(z)$$

for all $z \in U_1$. Moreover, it satisfies the assumptions of Proposition 4.11; therefore its restriction to $J(T_\alpha)$ must be the identity. In other words, $\psi_1 = \psi_2$ on $J(f)$.

□

In particular, if ψ_2 is sufficiently close to ψ_1 everywhere on U, then $\psi_1 = \psi_2$ on $J(f)$.

6. Tangent-like Families: J-stability and straightening

In this section we consider holomorphic families of tangent-like maps. Under certain conditions, each member of the family is hybrid equivalent to a member of the model family T_α, by the Straightening Theorem (Theorem A). This defines a straightening map χ between the connectedness locus of the TL family and the Tandelbrot set. Our goal in this section is to prove that this map is actually a homeomorphism, proving Theorem C.

The section is divided in three parts. Subsection 6.1 introduces TL families and fixes the conditions under which we can define χ, appart from discussing the family of hybrid equivalences that arise from the Straightening Theorem. In subsection 6.2, we discuss $J-$stability for TL families and show that the boundary of the connected locus is the bifurcation locus. Finnally, we use all our previous work to show in subsection 6.3 that the straightening map is a homeomoprhism and conclude the proof of Theorem C.

6.1. **Tangent-like families and definition of the straightening map.** The following definition mimics the definition in [Lyu24, p. 545].

Definition 6.1 (Tangent-like families)**.** Let $\Lambda \subset \mathbb{C}$ be a domain. A holomorphic family of TL maps is a family of TL maps of the form $f_\lambda : U_\lambda \to V_\lambda$, $\lambda \in \Lambda$, such that:

(1) $\mathcal{U} := \{(\lambda, z) : z \in U_\lambda\}$ and $\mathcal{V} := \{(\lambda, z) : z \in V_\lambda\}$ are domains in $\mathbb{C}^2$
(2) $\mathbf{f} : \mathcal{U} \to \hat{\mathbb{C}}$ defined by $\mathbf{f}(\lambda, z) = f_\lambda(z)$ is holomorphic.

As in the case of holomorphic families of quadratic-like maps, we will make additional assumptions. We require, for a start, that the fundamental annulus moves holomorphically with the parameter.

Definition 6.2 (Equipped TL family)**.** A holomorphic family of TL maps is *equipped* if there exists $\lambda_0 \in \Lambda$ and a holomorphic motion $h_\lambda : \overline{V_{\lambda_0}} \setminus U_{\lambda_0} \to \overline{V_\lambda} \setminus U_\lambda$ such that for all $\lambda \in \Lambda$, for all $z \in \partial U_{\lambda_0} \setminus \{u_{\lambda_0}\}$,

$$h_\lambda \circ f_\lambda(z) = f_\lambda \circ h_\lambda(z).$$

The holomorphic motion $(h_\lambda)_{\lambda \in \Lambda}$ will be called the *equivariant* holomorphic motion of the fundamental annulus.

Definition 6.3 (Proper TL family)**.** A holomorphic family of TL maps is *proper* if:

(1) Λ is a Jordan domain, and $\{f_\lambda\}_{\lambda \in \Lambda}$ is the restriction of a slightly larger TL family $\{f_\lambda\}_{\lambda \in \widetilde{\Lambda}}$, where $\Lambda \Subset \widetilde{\Lambda}$.
(2) For all $\lambda \in \partial\Lambda$, $v_\lambda \in \partial V_\lambda$.
(3) The winding number of $v_\lambda - u_\lambda$ is 1, as λ turns once along $\partial\Lambda$.

From now on, we will always work with a proper equipped TL family.

Definition 6.4 (The filled Julia set and the Tandelbrot set of a TL family)**.** The *Tandelbrot set of a TL family* is its connectedness locus

$$\mathcal{T}(\mathbf{f}) := \{\lambda \in \Lambda : v_\lambda \in K(f_\lambda)\} = \{\lambda \in \Lambda : K(f_\lambda) \text{ is connected}\}$$

where $K(f_\lambda)$ denotes the filled Julia set

$$K(f_\lambda) := \bigcap_{n\geq 0} \overline{f_\lambda^{-n}(U_\lambda)}.$$

Similarly to the case of families of quadratic-like map, a tubing will be a map $t : \overline{\mathcal{V}}\backslash\mathcal{U} \to \Lambda\times A$ of the form $t(\lambda, z) = (\lambda, t_\lambda(z))$, where A is some fixed annulus. We will not require a general definition of tubing, and instead we will work only with so-called standard tubings defined below.

Definition 6.5 (Standard choice of tubing)**.** Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a proper equipped TL family: by definition, there is a holomorphic motion h_λ of the fundamental annulus $\overline{V_\lambda}\backslash U_\lambda$, with basepoint $\lambda_* \in \Lambda$. Let $U, V \subset \hat{\mathbb{C}}$ be such that $(T_0, U, V, \infty, \infty)$ is a TL map, and let $A := V \setminus \overline{U}$. We choose a conformal isomorphism $\varphi : V_{\lambda_*} \setminus \overline{U_{\lambda_*}} \to A$: it extends continuously to $\overline{V_{\lambda_*}} \setminus U_{\lambda_*}$. We then define the tubing map

$$t(\lambda, z) := (\lambda, t_\lambda(z)) := (\lambda, \varphi \circ h_\lambda^{-1}(z)).$$

Observe that this tubing map may be extended to a homeomorphism $\widetilde{t} : \overline{\mathcal{V}} \to \overline{\Lambda} \times \overline{V}$, which maps $\mathcal{U}$ to $\Lambda \times U$. Indeed, first extend φ to a quasiconformal homeomorphism $\widetilde{\varphi} : V_{\lambda_*} \to V$, and then extend h_λ by Slodkowski's theorem to a holomorphic motion $\widetilde{h}_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and set for all $(\lambda, z) \in \mathcal{V}$:

$$\widetilde{t}(\lambda, z) = (\lambda, \widetilde{\varphi} \circ \widetilde{h}_\lambda^{-1}(z)).$$

The map $\widetilde{t}$ is clearly continuous and injective on the compact set $\overline{\mathcal{V}}$, hence it is a homeomorphism onto its image. By continuity, $\widetilde{h}_\lambda$ maps U_{λ_*} to U_λ, hence $\widetilde{t}(\lambda, \cdot)$ maps U_λ to U. In particular, for a proper equipped family of TL maps, the first condition in Definition 6.1 is automatically satisfied.

Proposition 6.6 (Straightening theorem in a family)**.** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a proper equipped TL family. Let t be the tubing map defined above. For every $\lambda \in \Lambda$, there is a unique hybrid conjugacy φ_λ between f_λ and some T_α, $\alpha \in \mathbb{D}$ such that*

$$\frac{\overline{\partial}\varphi_\lambda}{\partial\varphi_\lambda} = \frac{\overline{\partial}t_\lambda}{\partial t_\lambda}$$

a.e. on $V_\lambda \setminus U_\lambda$. We define the straightening map

$$\chi : \Lambda \to \mathbb{D}$$

by the relation

$$\varphi_\lambda \circ f_\lambda = T_{\chi(\lambda)} \circ \varphi_\lambda.$$

Moreover, $\chi_{|\mathcal{T}(\mathbf{f})}$ does not depend on the choice of the tubing.

Proof. The existence and unicity of the hybrid conjugacies φ_λ follows from an inspection of the proof of the Straightening Theorem: Indeed, for every TL map (f, U, V, u, v), the construction of the hybrid conjugacy is completely determined by the choice of a quasiconformal homeomorphism t between $\overline{V} \setminus U$ and $\overline{V'} \setminus U'$ such that $t \circ f = T_0 \circ t$ on $\partial U \setminus \{u\}$, where U', V' and Jordan domains such that $(T_0, U', V', \infty, \infty)$ is TL. In our case, this homeomorphism is the tubing map t_λ.

The fact that if $\lambda \in \mathcal{T}(\mathbf{f})$ then $\chi(\lambda)$ does not depend on the choice of the tubing map follows from the unicity in the Straightening Theorem. □

Naturally, the straightening map sens the Tandelbrot set of $\mathbf{f}$ to the actual Tandelbrot set.

Lemma 6.7. *For any $\lambda \in \Lambda$, we have: $\lambda \in \mathcal{T}(\mathbf{f}) \Leftrightarrow \chi(\lambda) \in \mathcal{T}$.*

Proof. Indeed, recall that $\lambda \in \mathcal{T}(\mathbf{f})$ if and only if $K(f_\lambda)$ is connected (see Prop. 3.7), and that $\chi(\lambda) \in \mathcal{T}$ if and only if $K(T_{\chi(\lambda)})$ is connected . The claim is then clear, as the hybrid conjugacy gives a homeomorphism between $K(f_\lambda)$ and $K(T_{\chi(\lambda)})$. □

In order to prove the continuity of χ, we will require the following result.

Proposition 6.8 (Convergence of conjugacies)**.** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a proper tangent-like family, $\lambda_n \to \lambda_0 \in \Lambda$ with $\lambda_n \in \mathcal{T}(\mathbf{f})$, $\alpha_n := \chi(\lambda_n)$, ψ_n be the hybrid conjugacies between f_{λ_n} and T_{α_n}. Then, up to passing to a subsequence, we can assume that $\alpha_{n_k} \to \alpha_*$ and that ψ_{n_k} converge to a quasiconformal conjugacy ψ_* between f_{λ_0} and T_{α_*}.*

Notice that ψ_* is not necessarily a hybrid conjugacy, since filled Julia sets do not always move continuously with the parameter.

Proof. Since the Tandelbrot set is compact, up to extracting a subsequence we can assume that $\alpha_n \to \alpha_*$. We now show that the maps ψ_n are uniformly quasiconformal. For each λ_n the quasiconformality constant of the hybrid conjugacy ψ_n depends only on the modulus of the annulus $A_n := V_{\lambda_n} \setminus \overline{U_{\lambda_n}}$. Indeed, the quasiconformality constant of the interpolating map ψ in the proof of the straightening theorem depends only on the modulus of such annulus by Lemma 2.5, and is not modified by any of the subsequent pullbacks. Since $\{f_\lambda\}_{\lambda\in\Lambda}$ is equipped, the moduli of the annuli A_n are close to the modulus of the annulus $V_{\lambda_o} \setminus U_{\lambda_0}$. So the maps ψ_n are uniformly quasiconformal.

By compactness of quasiconformal maps with the same quasiconformality constant, up to passing to a subsequence (ψ_n) converge to a quasiconformal limit map ψ_* uniformly on compact subsets of U_{λ_*}. Moreover, for each n we have

$$\psi_n \circ f_{\lambda_n} = T_{\alpha_n} \circ \psi_n.$$

Up to a passing to a subsequence, the left hand side converges to $\psi_* \circ f_{\lambda_0}$ uniformly on compact subsets of U_{λ_0} (we are using again that the annuli A_n move holomorphically), while the right hand side converges to $T_{\alpha_*} \circ \psi_*$, implying that ψ_* is a quasiconformal conjugacy between f_{λ_0} and T_{α_*}.

□

The following Proposition asserts that proper equipped TL families are natural families (see Section 2.3).

Proposition 6.9 (Proper equipped TL families are natural)**.** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a proper equipped TL family. Then it is a natural family, i.e. for any $\lambda_* \in \Lambda$, there exists holomorphic motions $\varphi_\lambda : V_{\lambda_*} \to V_\lambda$ and $\psi_\lambda : U_{\lambda_*} \to U_\lambda$ such that*

$$f_\lambda = \varphi_\lambda \circ f_{\lambda_*} \circ \psi_\lambda^{-1}.$$

Proof. Let $\lambda_* \in \Lambda$ be arbitrary. By the assumption that $\{f_\lambda\}_{\lambda\in\Lambda}$ is equipped, there exists a holomorphic motion $h_\lambda : \overline{V_{\lambda_*}} \setminus U_{\lambda_*} \to \overline{V_\lambda} \setminus U_\lambda$. Now consider the holomorphic motion $\varphi_\lambda : (\partial V_{\lambda_*}) \cup \{v_{\lambda_*}\} \to (\hat{\mathbb{C}} \setminus V_\lambda) \cup \{v_\lambda\}$ defined by

$$\varphi_\lambda(z) = \begin{cases} h_\lambda(z) & \text{if } z \in \partial V_{\lambda_*} \\ v_\lambda & \text{if } z = v_{\lambda_*}. \end{cases}$$

It is indeed a holomorphic motion, since for all $\lambda \in \Lambda$, $v_\lambda \in V_\lambda$. Now extend φ_λ using Slodkowski's λ-lemma ([Slo91]; compare [Lyu24], Fourth λ-Lemma) to obtain a holomorphic motion of the whole Riemann sphere, which we still denote by φ_λ. By construction, for any $\lambda \in \Lambda$, φ_λ maps V_{λ_*} to V_λ and v_{λ_*} to v_λ.

Next, let us again apply Slodkowski's λ-lemma to h_λ, to obtain an extension of h_λ to the whole Riemann sphere, which we denote by $\widetilde{\psi}_\lambda$. Let us emphasize that a priori, $\widetilde{\psi}_\lambda$ and φ_λ need not coincide outside of ∂V_λ. For any $\lambda \in \Lambda$, $\widetilde{\psi}_\lambda$ maps U_{λ_*} to U_λ. The rest of the proof is now similar to [[ABF21], Theorem 2.6]. We include it for the convenience of the reader.

Let $g_\lambda := \varphi_\lambda^{-1} \circ f_\lambda \circ \widetilde{\psi}_\lambda$. For every $\lambda \in \Lambda$, $g_\lambda : U_{\lambda_*} \to V_{\lambda_*} \setminus \{v_{\lambda_*}\}$ is a universal covering map. Let $\lambda \in \Lambda$ and let $\gamma : [0,1] \to \Lambda$ a continuous path joining λ_* to λ. By [[ABF21], Lemma 2.7], there exists a homotopy $\widetilde{k}_t : U_{\lambda_*} \to U_{\lambda_*}$ such that for all $t \in [0,1]$,

$$g_{\gamma(t)} = g_{\lambda_*} \circ \widetilde{k}_t.$$

Since Λ is simply connected, for every $z \in U_{\lambda_*}$ the value $\widetilde{k}_1(z)$ is independant from the choice of γ, and depends only on λ and z: we denote it by $k_\lambda(z)$. This gives the existence of a continuous map $(\lambda, z) \mapsto k_\lambda(z)$ such that for all $\lambda \in \Lambda$,

$$g_\lambda = g_{\lambda_*} \circ k_\lambda = f_{\lambda_*} \circ k_\lambda.$$

For every $\lambda \in \Lambda$, the map $k_\lambda : U_{\lambda_*} \to U_{\lambda_*}$ is a homeomorphism. Let $\psi_\lambda := \widetilde{\psi}_\lambda \circ k_\lambda^{-1}$: then the previous relation may be rewritten as

$$f_\lambda = \varphi_\lambda \circ f_{\lambda_*} \circ \psi_\lambda^{-1}.$$

It only remains to prove that ψ_λ defines a holomorphic motion, i.e. that for every $z \in U_{\lambda_*}$, $\lambda \mapsto \psi_\lambda(z)$ is holomorphic. This follows from the relation $f_\lambda \circ \psi_\lambda = \varphi_\lambda \circ f_{\lambda_*}$ and the Implicit Function Theorem applied to the holomorphic map $(\lambda, z) \mapsto f_\lambda(z) - \varphi_\lambda \circ f_{\lambda_*}(z)$. ☐

We also record here the following useful lemma, which will be used repeatedly in Section 6.3.

Lemma 6.10 (Lifting a holomorphic motion)**.** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a holomorphic family of TL maps. Let $\lambda_0 \in \Lambda$, $D \subset \Lambda$ be a simply connected domain containing λ_0 and $X \subset \overline{V_{\lambda_0}}$. Let $\varphi^{(0)} : D \times X \to \hat{\mathbb{C}}$ be a holomorphic motion with basepoint λ_0 such that for all $(\lambda, z) \in D \times f_{\lambda_0}^{-1}(X)$, $\varphi_\lambda^{(0)} \circ f_{\lambda_0}(z) \neq v_\lambda$. Then there exists a unique holomorphic motion $\varphi^{(1)} : D \times f_{\lambda_0}^{-1}(X) \to \hat{\mathbb{C}}$ such that for all $\lambda \in D$,*

$$f_\lambda \circ \varphi_\lambda^{(1)} = \varphi_\lambda^{(0)} \circ f_{\lambda_0}.$$

Proof. By Proposition 6.9, we may write $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$, where $\varphi_\lambda, \psi_\lambda$ are holomorphic motions of $\overline{V_\lambda}$ and $\overline{U_\lambda}$ respectively, with basepoint λ_0.

Let $z_0 \in f_{\lambda_0}^{-1}(X)$. By the lifting property for covering maps, there exists a unique lift γ_{z_0} by f_{λ_0} of the continuous map

$$D \ni \lambda \mapsto \varphi_\lambda^{-1} \circ \varphi_\lambda^{(0)} \circ f_{\lambda_0}(z_0),$$

such that $\gamma_{z_0}(\lambda_0) = z_0$. Indeed, φ_λ^{-1} maps v_λ to v_{λ_0}, and by assumption $\varphi_\lambda^{(0)} \circ f_{\lambda_0}(z_0) \neq v_\lambda$ for all $\lambda \in D$, so $\varphi_\lambda^{(0)} \circ f_{\lambda_0}(z_0) \in \overline{V_{\lambda_0}} \setminus \{v_{\lambda_0}\}$ for all $\lambda \in D$, and D is simply connected.

We then let $\varphi_\lambda^{(1)}(z_0) := \psi_\lambda(\gamma_{z_0}(\lambda))$. By construction, we have

$$f_\lambda \circ \varphi_\lambda^{(1)}(z_0) = \varphi_\lambda^{(0)} \circ f_{\lambda_0}(z_0)$$

and $\varphi_{\lambda_0}^{(1)}(z_0) = z_0$.

By the Implicit Function Theorem, $\lambda \mapsto \varphi_\lambda^{(1)}(z_0)$ is holomorphic on D. By the uniqueness of the lifting property, the map $z \mapsto \varphi_\lambda^{(1)}(z)$ is injective for all $\lambda \in D$. Thus, $\varphi^{(1)} : D \times f_{\lambda_0}^{-1}(X) \to \hat{\mathbb{C}}$ is indeed a holomorphic motion. $\square$

6.2. **J-stability in TL families.** In this section we define activity and passivity for the singular value v_λ of a tangent-like family, and we show that the activity locus of v_λ is $\partial\mathcal{T}(\mathbf{f})$.

Definition 6.11 (Activity and passivity of v_λ)**.** Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a proper equipped TL family, with asymptotic value v_λ. We say that v_λ is passive at $\lambda_0 \in \Lambda$ if one of the two following cases occur:

(1) there exists an open neighborhood $W \subset \Lambda$ of λ_0 such that for all $n \geq 0$, for all $\lambda \in W$, $f_\lambda^n(v_\lambda) \in U_\lambda$ and $\{\lambda \mapsto f_\lambda^n(v_\lambda) : n \in \mathbb{N}\}$ is a normal family;
(2) or there exists $n_0 \in \mathbb{N}$ such that $f_{\lambda_0}^n(v_{\lambda_0}) \in \overline{V_{\lambda_0}} \setminus \overline{U_{\lambda_0}}$.

We say that it is active at λ_0 if it is not passive. The activity locus is the set of $\lambda \in \Lambda$ such that v_λ is active at λ.

Remark 6.12 (Virtual cycles imply activity)**.** We say that f_λ has a virtual cycle if there exists $n \in \mathbb{N}$ such that $f_\lambda^n(v_\lambda) = \infty$. An important remark is that in a proper TL family, virtual cycles cannot be persistent. Indeed, otherwise we would have that $f_\lambda^n(v_\lambda) \equiv \infty$ in some open set D, which by the Identity Theorem implies that $f_\lambda^n(v_\lambda) \equiv \infty$ on Λ, which contradicts the properness assumption. It then follows from the definition that the existence of a virtual cycle implies activity of v_λ. Moreover, using Montel's theorem and the definition of activity, one can see that parameters with virtual cycles are dense in the activity locus. See Proposition 5.5 in [ABF21], observing that tangent-like maps always have non-omitted poles since they are hybrid equivalent to maps in the tangent family.

Proposition 6.13 ($J-$stability and activity locus)**.** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a TL family. Then $\{f_\lambda\}$ is J-stable near $\lambda_0 \in \Lambda$ if and only if v_λ is passive at λ_0. Moreover, the activity locus of v_λ is exactly $\partial\mathcal{T}(\mathbf{f})$.*

The proof uses the same type of arguments as in [ABF21], but will be simpler since we will not have to deal with persistent virtual cycles or critical points.

Proof. Let us first prove that the family $\{f_\lambda\}$ is J-stable near any parameter λ_0 in the passivity locus. Let $W \subset \Lambda$ be any simply connected neighborhood of λ_0 contained in the passivity locus. For any $n \in \mathbb{N}$, let

$$Z_n := \{(\lambda, z), \lambda \in W \text{ and } z \in U_\lambda : f_\lambda^n(z) = z\},$$

and let $\pi : Z_n \to W$ denote the projection on the first coordinate. The set Z_n is an analytic subset of the domain $\mathcal{U} \subset \mathbb{C}^2$. Since W is in the passivity locus, there cannot be nonpersistent parabolic parameters in W, so by the Implicit Function Theorem, Z_n is a smooth complex 1-manifold and $\pi : Z_n \to W$ is locally invertible (i.e.π has no critical values).

Also, by [ABF21, Lemma 3.4], a parameter $\lambda_* \in \Lambda$ is an asymptotic value for π only if the map f_{λ_*} has a virtual cycle of period at most n. Such virtual cycle cannot be persistent by Remark 6.12, and cannot be non-persistent because W is in the passivity locus. So π does not have asymptotic values either.

This proves that for any connected component Z of Z_n, $\pi : Z \to W$ is a covering map; but since W is simply connected, it is a conformal isomorphism; in other words, Z is a holomorphic graph above W. Since this is true for any connected component of Z_n, we have proved that all periodic points move holomorphically over W. By the λ-lemma and the density of repelling periodic points in $J(f_\lambda)$ (which follows easily from the Straightening Theorem), the family $\{f_\lambda\}_{\lambda \in W}$ is therefore J-stable.

Let us now prove that if W is a domain which intersects the activity locus, then $\{f_\lambda\}_{\lambda \in W}$ is not J-stable. By density of paraemeters with virtual cycles (see Remark 6.12), W contains a parameter λ_0 with a virtual cycle: $f^n_{\lambda_0}(v_{\lambda_0}) = \infty$. Assume for a contradiction that $\{f_\lambda\}_{\lambda \in W}$ is J-stable: then for all $\lambda \in W$, $f^n_\lambda(v_\lambda) = \infty$, or in other words, there is a persistent virtual cycle. But as observed in Remark 6.12, this is not possible for a proper TL family.

We now prove that the activity locus coincides with $\partial \mathcal{T}(\mathbf{f})$. Observe that for $\lambda \in \mathcal{T}(\mathbf{f})$, $v_\lambda \in K_\lambda$ hence $f^n_\lambda(v_n) \in U_\lambda$ for all $n \in \mathbb{N}$, while for $\lambda \in \Lambda \setminus \mathcal{T}(\mathbf{f})$, there is n such that $f^n_\lambda(v_\lambda) \in V_\lambda \setminus \overline{U_\lambda}$ (in particular, v_λ is passive for $\lambda \in \Lambda \setminus \mathcal{T}(\mathbf{f})$). So if $\lambda \in \partial \mathcal{T}(\mathbf{f})$, v_λ must be active at λ: it cannot satisfy (1) in Definition 6.11 because the family of iterates of the singular value is not well defined for parameters in $\Lambda \setminus \mathcal{T}(\mathbf{f})$, and it cannot satisfy (2) either because $f^n_\lambda(v_n) \in U_\lambda$ for all $n \in \mathbb{N}$. It remains to show that if λ_0 is in the interior of $\mathcal{T}(\mathbf{f})$ then v_λ is passive at λ_0. For all λ in a neighborhood D of λ_0 and all $n \in \mathbb{N}$, $f^n_\lambda(v_\lambda) \in U_\lambda$, so the family $(f^n_\lambda(v_\lambda))$ takes values in a bounded set when $\lambda \in D$, and normality follows by Montel's Theorem. □

Corollary 6.14. *Any parameter $\lambda \in \partial \mathcal{T}(\mathbf{f})$ is approximated by attracting parameters.*

Proof. Since $\mathcal{T}(\mathbf{f})$ is in the activity locus, and tangent-like maps have non-omitted poles by the straightening theorem, we can use Corollary 5.9 in [ABF21] to obtain the claim. □

6.3. **The straightening map is a homeomorphism.** In this section we aim at showing that for proper equipped tangent-like families, the straightening map χ induced by Proposition 6.6 is a homeomorphism between $\mathcal{T}(\mathbf{f})$ and $\mathcal{T}$. We first look at the strightening map on $\Lambda \setminus \mathcal{T}(\mathbf{f})$, to obtain a criterion for the injectivity of χ in terms of the winding number of $v_\lambda - u_\lambda$ around 0; we then look at χ on $\mathcal{T}(\mathbf{f})$, with the main goal of showing that fibers are discrete (see Proposition 6.24); we then use this result, together with a topological argument by Douady and Hubbard, to deduce that χ is a homeomorphism.

6.3.1. *The straightening map outside of $\mathcal{T}(\mathbf{f})$.* Our goal in this subsection is to prove the following statement.

Proposition 6.15 (Continuity in the disconnectedness locus)**.** *The straightening map χ and the hybrid conjugacy φ_λ both depend continuously on λ on $\Lambda \setminus \mathcal{T}(\mathbf{f})$.*

The next Lemma shows that it sufices to prove that the sequence of area functions

$$X_n(\lambda) := \text{area}\left(\bigcap_{0 \le k \le n} f_\lambda^{-k}(U_\lambda) \setminus K(f_\lambda) \right)$$

decrease uniformly to 0.

Lemma 6.16 (Area condition). *If the sequence of maps $\{X_n(\lambda)\}_{n\geq 0}$ converges locally uniformly to* 0 *outside $\mathcal{T}(\mathbf{f})$, then Proposition 6.15 holds.*

Proof. If we can prove that the hybrid conjugacy φ_λ depends continuously on λ, then the continuity of χ will follow, since $\chi(\lambda) = \frac{1}{\varphi_\lambda(v_\lambda)}$. Recall that φ_λ is the restriction to V_λ of a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, whose Beltrami coefficient we denote by μ_λ. To prove continuity of $\lambda \mapsto \varphi_\lambda$, it is enough to prove

(a) that there exists $0 < c < 1$ and $\delta > 0$ such that for all $\lambda \in \mathbb{D}(\lambda_0, \delta)$, $\|\mu_\lambda\|_{L^\infty} \leq c$;
(b) and that the map $\lambda \mapsto \mu_\lambda$ is continuous in the L^1 topology.

(See [DH85, I. 3)]). In order to prove (a) and (b), let us go back through the construction of φ_λ.

For all $\lambda \in \Lambda$, the tubing map t_λ is a quasiconformal homeomorphism between the annulus $V_\lambda \setminus U_\lambda$ and the annulus $V \setminus T_0^{-1}(V)$, where $T_0 : T_0^{-1}(V) \to V$ is TL. Moreover, t_λ is more precisely of the form $t_\lambda = t_{\lambda_0} \circ h_\lambda^{-1}$, where $h_\lambda : \overline{V_{\lambda_0}} \setminus U_{\lambda_0} \to \overline{V_\lambda} \setminus U_\lambda$ is the equivariant motion of the fundamental annulus. Let $\mu_{\lambda,0} := \frac{\overline{\partial} t_\lambda}{\partial t_\lambda}$ be the Beltrami form on $V_\lambda \setminus U_\lambda$, and for any $n \in \mathbb{N}$, let

$$\mu_{\lambda,n} := \begin{cases} (f_\lambda^j)^* \mu_{\lambda,0} & \text{on } f_\lambda^{-j}(V_\lambda \setminus U_\lambda) \text{ for } 0 \leq j \leq n \\ 0 & \text{elsewhere.} \end{cases}$$

By construction, $\mu_\lambda = \lim_{n\to\infty} \mu_{\lambda,n}$, pointwise a.e. Property (a) follow from the fact that $\|\mu_\lambda\|_{L^\infty} = \|\mu_{\lambda,0}\|_{L^\infty}$, which is locally bounded since $(h_\lambda)_{\lambda\in\Lambda}$ is a holomorphic motion. Let us now prove (b).

By the explicit expression of $\mu_{\lambda,0}$ in terms of $\mu_{\lambda_0,0}$, if $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of parameters with $\lim_{k\to+\infty} \lambda_k = \lambda_0$, then $\mu_{\lambda_k,0}$ converges pointwise a.e. to $\mu_{\lambda_0,0}$. Moreover, by definition of a quasiconformal map, we have $|\mu_{\lambda_k,0}(z)| < 1$ for all $k \in \mathbb{N}$ and for a.e. $z \in \hat{\mathbb{C}}$. By the dominated convergence theorem, it follows that $\lim_{k\to+\infty} \|\mu_{\lambda_k,0} - \mu_{\lambda_0,0}\|_{L^1} = 0$. In other words, $\lambda \mapsto \mu_{\lambda,0}$ is continuous in the L^1 topology. More generally, by a similar argument, for any fixed n, the map $\lambda \mapsto \mu_{\lambda,n}$ is also continuous in the L^1 topology. To ensure continuity of $\lambda \mapsto \mu_\lambda$, we must therefore prove that the sequence of maps $(\lambda \mapsto \mu_{\lambda,n})_{n\in\mathbb{N}}$ converges uniformly in $C^0(W, L^1(\hat{\mathbb{C}}))$, where W is some fixed neighborhood of λ_0; but observe that

$$\|\mu_{\lambda,n} - \mu_\lambda\|_{L^1} \leq \text{area}\left(\bigcap_{0\leq k\leq n} f_\lambda^{-k}(U_\lambda) \setminus K(f_\lambda)\right).$$

It is therefore indeed sufficient to prove that area $\left(\bigcap_{0\leq k\leq n} f_\lambda^{-k}(U_\lambda) \setminus K(f_\lambda)\right)$ converges uniformly to 0 on a neighborhood W of λ_0. □

The next lemma says that V_λ moves locally holomorphically, with a holomorphic motion that preserves the dynamics.

Lemma 6.17 (Quasiconformal conjugacies). *Let $\lambda_0 \in \Lambda \setminus \mathcal{T}$. There exists a neighborhood W of λ_0 and a holomorphic motion $H_\lambda : V_{\lambda_0} \to V_\lambda$ ($\lambda \in W$) such that for all $z \in U_{\lambda_0}$,*

$$H_\lambda \circ f_{\lambda_0}(z) = f_\lambda \circ H_\lambda(z).$$

Proof of Lemma 6.17. Let $\lambda_0 \notin \mathcal{T}$: then there exists $N \in \mathbb{N}$ such that $f_{\lambda_0}^N(v_{\lambda_0}) \in V_{\lambda_0} \setminus (U_{\lambda_0} \cup \{u_{\lambda_0}\})$.

If $f_{\lambda_0}^N(v_{\lambda_0}) \in \partial U_{\lambda_0} \setminus \{u_{\lambda_0}\}$, then up to reducing slightly V_{λ_0}, we may reduce to the case where $f_{\lambda_0}^N(v_{\lambda_0}) \in V_{\lambda_0} \setminus \overline{U_{\lambda_0}}$, which we assume from now on. Let D be a small simply connected neighborhood of λ_0 such that for all $\lambda \in D$, $f_\lambda^N(v_\lambda) \in V_\lambda \setminus \overline{U_\lambda}$. Let $(h_\lambda)_{\lambda \in D}$ denote the equivariant holomorphic motion of $\overline{V_\lambda} \setminus U_\lambda$ given by the definition of a holomorphic family of TL maps: we will modify h_λ so that $h_\lambda(f_{\lambda_0}^N(v_{\lambda_0})) = f_\lambda^N(v_\lambda)$. To this end, let $h_\lambda^{(0)}$ denote the holomorphic motion of $\partial V_{\lambda_0} \cup \partial U_{\lambda_0} \cup \{f_{\lambda_0}^N(v_{\lambda_0})\}$ over D defined by

$$h_\lambda(z) = \begin{cases} h_\lambda^{(0)}(z) & \text{if } z \in \partial V_{\lambda_0} \cup \partial U_{\lambda_0} \\ f_\lambda^N(v_\lambda) & \text{if } z = f_{\lambda_0}^N(v_{\lambda_0}) \ . \end{cases}$$

By Slodkowski's theorem, it may be extended to a holomorphic motion of $\hat{\mathbb{C}}$; and by continuity, this extension must map $\overline{V_{\lambda_0}} \setminus U_{\lambda_0}$ to $\overline{V_\lambda} \setminus U_\lambda$. We still denote by h_λ this extended holomorphic motion over D of $\overline{V_{\lambda_0}} \setminus U_{\lambda_0}$.

Applying successively Lemma 6.10, we obtain holomorphic motions $h^{(j)}$ (for $0 \leq j \leq N$) of $f_\lambda^{-j}(\overline{V_\lambda} \setminus U_\lambda)$ over D, such that

$$f_\lambda \circ h_\lambda^{(j+1)} = h_\lambda^{(j)} \circ f_{\lambda_0}.$$

For $j = N+1$, observe that since $h_\lambda^{(N)}(v_{\lambda_0}) = v_\lambda$ and since v_λ is an omitted value, we have $h_\lambda^{(N)} \circ f_{\lambda_0}(z) \neq v_\lambda$ for all $z \in f_{\lambda_0}^{-N-1}(\overline{V_{\lambda_0}} \setminus U_{\lambda_0})$. We may therefore still apply Lemma 6.10 to construct $h_\lambda^{(N+1)}$ on $f_{\lambda_0}^{-N-1}(\overline{V_{\lambda_0} \setminus U_{\lambda_0}})$, and more generally $h_\lambda^{(j)}$ for all $j \geq N+1$.

Moreover, since $h_\lambda^{(0)}$ is equivariant and by unicity of the lift, the $h_\lambda^{(j)}$ glue together continuously, i.e. we have $h_\lambda^{(j+1)} = h_\lambda^{(j)}$ on $f_\lambda^{-j}(\partial U_{\lambda_0})$. Therefore, for every $\lambda \in D$, the map H_λ defined by

$$H_\lambda(z) := h_\lambda^{(j)}(z) \text{ if } z \in f_{\lambda_0}^{-j}(\overline{V_{\lambda_0}} \setminus U_{\lambda_0})$$

is a quasiconformal homeomorphism satisfying

$$f_\lambda \circ H_\lambda(z) = H_\lambda \circ f_{\lambda_0}(z)$$

for all $z \in \bigcup_{j \geq 1} f_{\lambda_0}^{-j}(\overline{V_{\lambda_0}} \setminus U_{\lambda_0}) = U_{\lambda_0} \setminus K(f_{\lambda_0})$. It is also a holomorphic motion of $\bigcup_{j \in \mathbb{N}} f_{\lambda_0}^{-j}(\overline{V_{\lambda_0}} \setminus U_{\lambda_0}) = V_{\lambda_0} \setminus K(f_{\lambda_0})$ over D, which extends by Slodkowski's theorem to a holomorphic motion of V_{λ_0} (here, we use the fact that $J(f_\lambda) = K(f_\lambda)$ has no interior if $\lambda \notin \mathcal{T}$). By continuity, we still have

$$f_\lambda \circ H_\lambda(z) = H_\lambda \circ f_{\lambda_0}(z)$$

for all $z \in U_{\lambda_0}$, which finishes the proof. □

We are finally ready to conclude the proof of continuity of the straightening map.

Proof of Proposition 6.15. Let $X_n(\lambda) := \bigcap_{0 \leq k \leq n} f_\lambda^{-k}(U_\lambda) \setminus K(f_\lambda)$. We have:

$$\begin{aligned} \operatorname{area}(X_n(\lambda)) &= \operatorname{area} H_\lambda\left(X_n(\lambda_0)\right) \\ &= \int_{X_n(\lambda_0)} \operatorname{Jac} H_\lambda(z) |dz|^2. \end{aligned}$$

Up to reducing a little bit the set W given by the lemma above, we may assume without loss of generality that there exists a constant $K > 0$ such that for all $\lambda \in W$, the quasiconformal distortion of H_λ is less than K. Recall that since $\operatorname{Jac} H_\lambda = |\partial H_\lambda|^2 - |\overline{\partial} H_\lambda|^2$ and $\|dH_\lambda\| = |\partial H_\lambda| + |\overline{\partial} H_\lambda|$, there exists a constant $C = C(K) > 1$ (explicit) such that for all $\lambda \in W$,

$$C^{-1}\|dH_\lambda\|^2 \leq \operatorname{Jac} H_\lambda \leq C\|dH_\lambda\|^2. \tag{6.1}$$

Let $N_n(\lambda) := \int_{X_n(\lambda_0)} \|dH_\lambda(z)\|^2 |dz|^2$. By the Measurable Riemann Mapping Theorem, the for a.e. z, the maps $\lambda \mapsto \partial H_\lambda(z)$ and $\lambda \mapsto \overline{\partial} H_\lambda(z)$ are holomorphic; therefore N_n is subharmonic. For every fixed $\lambda \in W$, the map $z \mapsto \operatorname{Jac} H_\lambda(z)$ is integrable, therefore by (6.1), so is the map $z \mapsto \|dH_\lambda(z)\|^2$. Since the $X_n(\lambda)$ are decreasing and their intersection is empty, it follows that N_n converges pointwise a.e. to 0.

Let $\epsilon > 0$ be small enough that $\mathbb{D}(\lambda_0, 2\epsilon) \subset W$. Then, since N_n is subharmonic, we have for all $\lambda \in \mathbb{D}(\lambda_0, \epsilon)$:

$$N_n(\lambda) \leq \frac{1}{\pi\epsilon^2} \int_{\mathbb{D}(\lambda,\epsilon)} N_n(u) \cdot |du|^2. \tag{6.2}$$

The sequence $(N_n)_{n\in\mathbb{N}}$ is pointwise decreasing, N_0 is integrable and $(N_n)_{n\in\mathbb{N}}$ converges a.e. to 0. Therefore, by the dominated convergence theorem, $\lim_{n\to\infty} \int_{\mathbb{D}(\lambda_0,2\epsilon)} N_n(u) \cdot |du|^2 = 0$. By (6.1) and (6.2), we therefore finally have, for all $\lambda \in \mathbb{D}(\lambda_0, \epsilon)$:

$$\int_{X_n(\lambda_0)} \operatorname{Jac} H_\lambda(z) |dz|^2 \leq \frac{1}{\pi\epsilon^2} \int_{\mathbb{D}(\lambda_0,2\epsilon)} N_n(u) \cdot |du|^2 \to_{n\to+\infty} 0.$$

This proves that $\lim_{n\to\infty} \sup_{\lambda\in\mathbb{D}(\lambda_0,\epsilon)} \operatorname{area}(X_n(\lambda)) = 0$, and we are done by Lemma 6.16. $\square$

6.3.2. *Regularity of the straightening map on $\mathcal{T}(\mathbf{f})$.* We now proceed to study the regularity of χ in different parts of $\mathcal{T}(\mathbf{f})$.

We start by showing that χ is continuous on the boundary of $\mathcal{T}(\mathbf{f})$ (c.f. [DH85, Prop. 14, p.313]).

Lemma 6.18 (Continuity on the boundary)**.** *The straightening map χ is continuous at every point $\lambda \in \partial\mathcal{T}(\mathbf{f})$, and $\chi(\partial\mathcal{T}(\mathbf{f})) \subset \partial\mathcal{T}$.*

Proof. Let $\lambda_* \in \partial\mathcal{T}(\mathbf{f})$: let us first prove that $\chi(\lambda_*) \in \partial\mathcal{T}$. By Remark 6.12, there exists a sequence $\lambda_n \to \lambda_*$ such that f_{λ_n} has a virtual cycle. Let $\alpha_n := \chi(\lambda_n)$: up to extracting a subsequence, we may assume that $\alpha_n \to \alpha \in \partial\mathcal{T}$ (indeed, $\partial\mathcal{T}$ is closed and $\alpha_n \in \partial\mathcal{T}$ since T_{α_n} also has a virtual cycle, as this property is invariant under topological conjugacy). Moreover, by the definition of χ, there is a uniformly quasiconformal sequence of hybrid conjugacies φ_n between f_{λ_n} and T_{α_n}. By Proposition 6.8, up to passing to a subsequence we may also assume that $(\varphi_n)_{n\in\mathbb{N}}$ converges locally uniformly to a quasiconformal homeomorphism φ conjugating f_{λ_*} to T_α on a neighborhood of their respective Julia sets. On the other hand, f_{λ_*} is also hybrid equivalent to $T_{\chi(\lambda_*)}$; therefore, T_α and $T_{\chi(\lambda_*)}$ are quasiconformal conjugated on a neighborhood of their respective Julia sets. By Corollary 4.30, the maps are quasiconformal conjugated on $\hat{\mathbb{C}}$. By Proposition 4.31, quasiconformal conjugacy classes in the model family are either trivial or open, and in particular, parameters in $\partial\mathcal{T}$ are rigid, i.e. they are alone in their quasiconformal conjugacy class. This proves that $\chi(\lambda_*) = \alpha$, and so $\chi(\lambda_*) \in \partial\mathcal{T}$, as announced.

Let us now prove that χ is continuous at λ_*. The reasoning is exactly similar, except that now we take an arbitrary sequence $\lambda_n \to \lambda_*$, and we must still prove that $\chi(\lambda_n) \to \chi(\lambda_*)$.

Reasoning as above, we obtain again (up to passing to a subsequence) that $\chi(\lambda_n) \to \alpha$ and that T_α and $T_{\chi(\lambda_*)}$ are globally quasiconformally conjugated, except that now we do not know a priori that $\alpha \in \partial\mathcal{T}$. But we have proved that $\chi(\lambda_*) \in \partial\mathcal{T}$, so we can still conclude that $\alpha = \chi(\lambda_*)$. Since the sequence $(\lambda_n)_{n\in\mathbb{N}}$ was arbitrary, we have proved that χ is continuous at λ_*. □

Lemma 6.19 (Continutity on the interior). *The straightening map χ is continuous on $\mathring{\mathcal{T}}(\mathbf{f})$.*

Proof. Let $\lambda_* \in \mathring{\mathcal{T}}(\mathbf{f})$ and $\lambda_n \to \lambda_*$. We will prove that $\chi(\lambda_n) \to \chi(\lambda_*)$. By compactness of $\mathcal{T}$, it is sufficient to prove that any limit of a subsequence of $\chi(\lambda_n)$ is equal to $\chi(\lambda_*)$: let α denote such a limit.

Then there is a subsequence $n_k \to +\infty$ such that for all $k \in \mathbb{N}$, $\varphi_k \circ f_{\lambda_{n_k}} = T_{\chi(\lambda_{n_k})} \circ \varphi_k$, for some uniformly quasiconformal sequence of homeomorphisms φ_k such that $\overline{\partial}\varphi_k = 0$ a.e. on $K(f_{\lambda_{n_k}})$. Up to extracting a further subsequence (see Proposition 6.8), we may assume without loss of generality that $\varphi_k \to \varphi$ quasiconformal homeomorphism such that

$$\varphi \circ f_{\lambda_*} = T_\alpha \circ \varphi.$$

But since $\lambda_* \in \mathring{\mathcal{T}}(\mathbf{f})$, Proposition 6.13 implies that $K(f_\lambda)$ moves holomorphically near λ_*. Passing to the limit, this implies that $\overline{\partial}\varphi = 0$ a.e. on $K(f_{\lambda_*})$: in other words, T_α is hybrid equivalent to f_{λ_*}. But so is $T_{\chi(\lambda_*)}$, so by unicity in the Straightening Theorem, $\alpha = \chi(\lambda_*)$ and we are done. □

In fact we see below that it maps hyperbolic components holomorphically and properly onto hyperbolic components.

Proposition 6.20 (Holomorphic and proper in hyperbolic components). *Let U be a hyperbolic component of $\mathcal{T}(\mathbf{f})$. The straightening map $\chi_{|U}$ is holomorphic and maps U properly into a hyperbolic component V of $\mathcal{T}$.*

Proof. By Lemmas 6.18 and 6.19, there is a component V of $\mathring{\mathcal{T}}$ such that $\chi(U) \subset V$ and $\chi : U \to V$ is continuous. Clearly, V is a hyperbolic component (since the property of having an attracting cycle is preserved under hybrid equivalence). Moreover, the multiplier is invariant under hybrid equivalence, so we have $\rho_V = \rho_U \circ \chi$, where $\rho_U : U \to \mathbb{D}^*$ and $\rho_V : V \to \mathbb{D}^*$ are the multiplier maps on U and V respectively.

By [FK21], the map ρ_V is a holomorphic universal cover; in particular, it is locally invertible. By the Implicit Function Theorem, the map $\rho_U : U \to \mathbb{D}^*$ is holomorphic. Therefore, we may locally write $\chi = \rho_V^{-1} \circ \rho_U$, which implies that χ is holomorphic on U.

By Lemma 6.18, the map χ is moreover continuous on $\overline{U}$, and $\chi(\partial U) \subset \partial\mathcal{T} \cap \overline{V} = \partial V$. Therefore, $\chi : U \to V$ is proper. □

To end this section, we prove that the same properties are true in the case that non-hyperbolic components existed. Our goal is to show the following statement (c.f. [Lyu24, Lemma 42.13 p. 553]).

Proposition 6.21 (Holomorphic and proper in non-hyperbolic components). *Let U be a non-hyperbolic component of $\mathcal{T}(\mathbf{f})$. The straightening map $\chi_{|U}$ is holomorphic and maps U properly into a non-hyperbolic component V of $\mathcal{T}$.*

First we need the following lemma (c.f. [Lyu24, Lemma 42.12 p. 553]).

Lemma 6.22. *Let λ_0 be in a non-hyperbolic component $Q \subset \mathcal{T}(\mathbf{f})$. Then there exists a neighborhood W of λ_0 and a holomorphic motion $H : W \times V_{\lambda_0} \to \hat{\mathbb{C}}$ such that for all $\lambda \in W$, for all $z \in U_{\lambda_0}$,*

$$H_\lambda \circ f_{\lambda_0}(z) = f_\lambda \circ H_\lambda(z).$$

Proof of Lemma 6.22. Let W be any simply connected domain contained in the non-hyperbolic component Q and containing λ_0. By assumption, there is a holomorphic motion $h_\lambda : \overline{V_{\lambda_0}} \setminus U_{\lambda_0} \to \overline{V_\lambda} \setminus U_\lambda$ such that for all $z \in \partial U_{\lambda_0} \setminus \{\infty\}$,

$$h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda.$$

Let $X_n(\lambda) := f_\lambda^{-n}(\overline{V_\lambda} \setminus U_\lambda)$. Observe that the interior of the $X_n(\lambda)$ are pairwise disjoint, and that

$$\bigcup_{n \geq 0} X_n(\lambda) = \overline{V_\lambda} \setminus K(f_\lambda).$$

By Lemma 6.10, there exists a unique lift $h_\lambda^{(1)}$ to $X_1(\lambda)$ of the holomorphic motion h_λ, satisfying

$$f_\lambda \circ h_\lambda^{(1)} = f_{\lambda_0} \circ h_\lambda.$$

But since by assumption h_λ is equivariant, i.e. $f_\lambda \circ h_\lambda(z) = f_{\lambda_0} \circ h_\lambda(z)$ for all $z \in \partial U_{\lambda_0} \setminus \{\infty\}$, we have in fact $h_\lambda^{(1)}(z) = h_\lambda(z)$ for all $z \in \partial U_{\lambda_0} \setminus \{\infty\}$. Additionnally, $h_\lambda^{(1)}$ extends to $\overline{X_1(\lambda)}$ by the λ-lemma, with $h_\lambda^{(1)}(\infty) = \infty$. In other words, h_λ and $h_\lambda^{(1)}$ coincide on their common boundary. Moreover, by definition h_λ maps $X_0(\lambda_0)$ to $X_0(\lambda)$, and $h_\lambda^{(1)}$ maps $X_1(\lambda_0)$ to $X_1(\lambda)$, whose interior are pairwise disjoint; therefore, for all $\lambda \in W$, the following map is well-defined and injective:

$$H_\lambda(z) = \begin{cases} h_\lambda(z) & \text{if } z \in X_0(\lambda) \\ h_\lambda^{(1)}(z) & \text{if } z \in X_1(\lambda) \end{cases}$$

It is also clearly holomorphic in λ, hence it defines a holomorphic motion. By construction, we have $H_\lambda \circ f_{\lambda_0}(z) = f_\lambda \circ H_\lambda(z)$ for all $z \in X_1(\lambda)$.

Proceeding in a similar way, we define inductively $h_\lambda^{(n+1)}$ as a lift of $h_\lambda^{(n)}$ using Lemma 6.10. This is possible since for all $\lambda \in W$, $v_\lambda \in K(f_\lambda)$. We also define $H_\lambda(z) := h_\lambda^{(n)}(z)$ for all $z \in X_n(\lambda)$ and all $\lambda \in W$. Arguing as above, this defines a holomorphic motion on W, and for all $(\lambda, z) \in W \times U_{\lambda_0} \setminus K(f_{\lambda_0})$,

$$f_\lambda \circ H_\lambda(z) = f_{\lambda_0} \circ H_\lambda(z).$$

□

Proof of Proposition 6.21. Let $\lambda_0 \in U$, and let W, H_λ be given by Lemma 6.22. Up to restricting if necessary, assume that $W \subset U$ and that W is simply connected.

Let $\mu_\lambda := \frac{\overline{\partial} H_\lambda}{\partial H_\lambda}$: then $f_{\lambda_0}^* \mu_\lambda = \mu_\lambda$ and $\lambda \mapsto \mu_\lambda$ is holomorphic. Let $\nu_\lambda(z) := \varphi_* \mu_\lambda(z)$ for $z \in J(T_{\chi(\lambda_0)})$ and 0 outside, where φ is the hybrid conjugacy between f_{λ_0} and $T_{\chi(\lambda_0)}$. Then $\nu_\lambda = T_{\chi(\lambda_0)}^* \nu_\lambda$.

Moreover, since φ is complex differentiable a.e. on $J(f_{\lambda_0})$, we have

$$\nu_\lambda = \mu_\lambda \circ \varphi^{-1} \cdot \frac{\varphi'}{\overline{\varphi'}} \circ \varphi^{-1}$$

and so $\lambda \mapsto \nu_\lambda$ is holomorphic. Let h_λ be the quasiconformal homeomorphisms which integrate ν_λ, normalized so that h_λ fix $0, 1, \infty$. Then $h_\lambda \circ T_{\chi(\lambda_0)} \circ h_\lambda^{-1}$ depends holomorphically on λ, is holomorphic with respect to z, and by Lemma 4.2, it is of the form $h_\lambda \circ T_{\chi(\lambda_0)} \circ h_\lambda^{-1} = T_{\alpha(\lambda)}$, for some holomorphic map $\alpha : W \to \mathbb{C}$.

Finally, observe that by construction, $T_{\alpha(\lambda)}$ is hybrid conjugated to f_λ via the quasiconformal homeomorphism $h_\lambda \circ \varphi \circ H_\lambda^{-1}$. Hence, by unicity of the straightening, $\alpha(\lambda) = \chi(\lambda)$ for all $\lambda \in W$. This proves that χ is holomorphic on the non-hyperbolic component U. □

6.3.3. *Finiteness of the fibers of χ above $\mathcal{T}$.* A crucial part of the proof of bijectivity of $\chi : \mathcal{T}(\mathbf{f}) \to \mathcal{T}$ is to show that its fibers are discrete (and hence finite, by compactness). We will use the following preliminary result, that connects the equivariant motion of the fundamental annulus with the hybrid conjugacies between the maps in the TL-family and the members of the model family.

Proposition 6.23 (Matching hybrid conjugacies with external homeomorphisms)**.** *Let $\lambda_0 \in \partial\mathcal{T}(\mathbf{f})$, such that $u_{\lambda_0} \neq v_{\lambda_0}$. Let h_λ denote the equivariant holomorphic motion of $\overline{V_\lambda} \setminus U_\lambda$, with basepoint λ_0. There exists $\epsilon_0 > 0$ such that if $\lambda \in \mathbb{D}(\lambda_0, \epsilon_0)$ and $\chi(\lambda) = \chi(\lambda_0)$, then h_λ extends to a quasiconformal homeomorphism $h_\lambda : V_{\lambda_0} \to V_\lambda$ such that for all $z \in U_{\lambda_0}$, we have*

$$f_\lambda \circ h_\lambda(z) = h_\lambda \circ f_{\lambda_0}(z)$$

and moreover h_λ coincides with $\varphi_\lambda \circ \varphi_{\lambda_0}^{-1}$ on $J(f_{\lambda_0})$, where $\varphi_\lambda, \varphi_{\lambda_0}$ denote the hybrid conjugacies associated to λ and λ_0 respectively.

Proof. Let $\alpha := \chi(\lambda_0)$, and let $\lambda_1 \in \Lambda$ such that $\chi(\lambda_1) = \chi(\lambda_0)$. We have the following commuting diagram:

$$\begin{tikzcd}
\varphi_{\lambda_1}(\partial U_{\lambda_1}) \setminus \{\infty\} \arrow[r, "T_\alpha"] & \varphi_{\lambda_1}(\partial V_{\lambda_1}) \\
\partial U_{\lambda_1} \setminus \{u_{\lambda_1}\} \arrow[u, "\varphi_{\lambda_1}"] \arrow[r, "f_{\lambda_1}"] & \partial V_{\lambda_1} \arrow[u, "\varphi_{\lambda_1}"'] \\
\partial U_{\lambda_0} \setminus \{u_{\lambda_0}\} \arrow[u, "h_{\lambda_1}"] \arrow[r, "f_{\lambda_0}"] \arrow[d, "\varphi_{\lambda_0}"'] & \partial V_{\lambda_0} \arrow[u, "h_{\lambda_1}"'] \arrow[d, "\varphi_{\lambda_0}"] \\
\varphi_{\lambda_0}(\partial U_{\lambda_0}) \setminus \{\infty\} \arrow[r, "T_\alpha"] & \varphi_{\lambda_0}(\partial V_{\lambda_0})
\end{tikzcd}$$

Consider the map $\Phi_{\lambda_1} := \varphi_{\lambda_1} \circ h_{\lambda_1} \circ \varphi_{\lambda_0}^{-1} : \varphi_{\lambda_0}(\overline{V_{\lambda_0}} \setminus U_{\lambda_0}) \to \varphi_{\lambda_1}(\overline{V_{\lambda_1}} \setminus U_{\lambda_1})$. By the diagram above,

$$T_\alpha \circ \Phi_{\lambda_1}(z) = \Phi_{\lambda_1} \circ T_\alpha(z)$$

for all $z \in \partial U_{\lambda_0} \setminus \{u_{\lambda_0}\}$. By Lemma 4.33, Φ_{λ_1} extends to a homeomorphism $\widetilde{\Phi}_{\lambda_1} : \varphi_{\lambda_0}(\overline{V_{\lambda_0}}) \setminus K(T_\alpha) \to \varphi_{\lambda_0}(\overline{V_{\lambda_0}}) \setminus K(T_\alpha)$ which commutes with T_α on $\varphi_{\lambda_0}(U_{\lambda_0}) \setminus K(T_\alpha)$.

Let p be the marked pole of T_α (see Lemma 4.7), and let $p_\lambda := \varphi_\lambda^{-1}(p)$. Let g_{λ_0} denote the inverse branch of f_{λ_0} mapping ∞ to p_{λ_0}, and let g_λ be the corresponding inverse branch for f_λ. There exists $\delta > 0$ such that for all λ sufficiently close to λ_0, g_λ is well-defined on $\mathbb{D}(u_{\lambda_0}, 2\delta)$, and $\lambda \mapsto g_\lambda(z)$ is holomorphic for all $z \in \mathbb{D}(u_{\lambda_0}, 2\delta)$.

Let $\mathcal{N} \subset \varphi_{\lambda_0}(f_{\lambda_0}^{-1}(V_{\lambda_0} \setminus U_{\lambda_0})) = T_\alpha^{-1}(\varphi_{\lambda_0}(V_{\lambda_0} \setminus U_{\lambda_0}))$ be a domain such that $p \in \partial\mathcal{N}$, and such that $\mathcal{N}$ is compactly contained in $\varphi_{\lambda_0}(U_{\lambda_0})$. This is possible since

$$p_{\lambda_0} \in f_{\lambda_0}^{-1}(\partial U_{\lambda_0}) \subset U_{\lambda_0}.$$

Moreover, up to reducing $\mathcal{N}$ if necessary, we may assume that for all λ close enough to λ_0,

$$h_\lambda \circ f_{\lambda_0} \circ \varphi_{\lambda_0}^{-1}(\mathcal{N}) \subset \mathbb{D}(u_{\lambda_0}, \delta).$$

Observe that this expression is well-defined not just for $\lambda = \lambda_1$ but for all λ close enough to λ_0, since by construction $f_\lambda \circ h_{\lambda_0}^{-1}(\mathcal{N}) \subset V_{\lambda_0} \setminus U_{\lambda_0}$.

Let $\epsilon > 0$ be the constant given by Proposition 4.11. We will prove that there exists $\epsilon_0 > 0$ such that if $|\lambda_1 - \lambda_0| < \epsilon_0$, then $\sup_{z\in\mathcal{N}} |\widetilde{\Phi}_{\lambda_1}(z) - z| < \epsilon$.

Recall that the hybrid conjugacies φ_λ depend continuously on λ at λ_0 (Lemma 6.18). By some diagram chasing, we may write, for $z \in \mathcal{N}$:

$$\widetilde{\Phi}_{\lambda_1}(z) = \varphi_{\lambda_1} \circ g_{\lambda_1} \circ h_{\lambda_1} \circ f_{\lambda_0} \circ \varphi_{\lambda_0}^{-1}(z) = (\varphi_{\lambda_1} \circ g_{\lambda_1}) \circ h_{\lambda_1} \circ \left(\varphi_{\lambda_0} \circ g_{\lambda_0}^{-1}\right)^{-1}(z). \tag{6.3}$$

As noted above, this expression is defined not just for $\lambda = \lambda_1$ but for all λ close enough to λ_1. Moreover, since $\varphi_\lambda, g_\lambda, h_\lambda$ all depend continuously on λ (uniformly in z), the map

$$\lambda \mapsto \varphi_\lambda \circ g_\lambda \circ h_\lambda \circ f_{\lambda_0} \circ \varphi_{\lambda_0}^{-1}$$

depends continuously on λ, uniformly on $z \in \mathcal{N}$. Since

$$\varphi_{\lambda_0} \circ g_{\lambda_0} \circ h_{\lambda_0} \circ f_{\lambda_0} \circ \varphi_{\lambda_0}^{-1} = \mathrm{Id},$$

it follows that at there exists $\epsilon_0 > 0$ such that if $|\lambda_1 - \lambda_0| < \epsilon_0$, then $\sup_{z\in\mathcal{N}} |\widetilde{\Phi}_{\lambda_1}(z) - z| < \epsilon$. We can then apply Proposition 4.11: the map $\widetilde{\Phi}_{\lambda_1}$ extends continuously to $J(T_\alpha)$ by the identity, hence it extends continuously to $K(T_\alpha)$ by the identity. The desired extension of h_{λ_1} is then given by

$$\widetilde{h}_{\lambda_1}(z) := \begin{cases} \varphi_{\lambda_1}^{-1} \circ \widetilde{\Phi}_{\lambda_1} \circ \varphi_{\lambda_0}(z) & \text{if } z \in \overline{V_{\lambda_0}} \setminus K(T_{\lambda_0}) \\ = \varphi_{\lambda_1}^{-1} \circ \varphi_{\lambda_0}(z) & \text{if } z \in K(f_{\lambda_0}). \end{cases}$$

□

We can now prove that the fibers of χ are finite (c.f. [Lyu24, Lemma 42.15, p. 554].

Proposition 6.24 (Finiteness of fibers). *The fibers of $\chi : \mathcal{T}(\mathbf{f}) \to \mathcal{T}$ are finite.*

Proof. By compactness of $\mathcal{T}(\mathbf{f})$, it is enough to prove that fibers are discrete. We first deal with the case where T_α has a virtual cycle: then there exists $n \in \mathbb{N}$ such that for any $\lambda \in \chi^{-1}(\{\alpha\})$, $f_\lambda^n(v_\lambda) = u_\lambda$; and since $\lambda \mapsto f_\lambda^n(v_\lambda)$ is holomorphic and non-constant (by the definition of a proper TL family), $\chi^{-1}(\{\alpha\})$ is discrete.

From now on, we therefore assume that T_α has no virtual cycle. Assume that there exists $\alpha \in \mathcal{T}$ such that $\chi^{-1}(\{\alpha\})$ is infinite: then there exists an injective sequence $\lambda_n \in \mathcal{T}(\mathbf{f})$ such that $\chi(\lambda_n) = \alpha$.

Up to extracting a subsequence, we may assume that it converges to some $\lambda_\infty \in \mathcal{T}(\mathbf{f})$. Moreover, since χ is holomorphic on $\mathring{\mathcal{T}}(\mathbf{f})$ (Propositions 6.20 and 6.21), we may assume without loss of generality that $\lambda_\infty \in \partial\mathcal{T}(\mathbf{f})$. Since χ is continuous on $\mathcal{T}(\mathbf{f})$ (Lemmas 6.18 and 6.19), $\chi(\lambda_\infty) = \alpha$, so f_{λ_∞} is hybrid equivalent to f_{λ_n} for all n. Let

$$h_\lambda : \overline{V_{\lambda_\infty}} \setminus U_{\lambda_\infty} \to \overline{V_\lambda} \setminus U_\lambda$$

be the holomorphic motion given by the definition of an equipped TL family with the choice of λ_∞ as a base point. Recall that h_λ conjugates f_{λ_∞} to f_λ on $\partial U_{\lambda_\infty} \setminus \{u_{\lambda_\infty}\}$. Let us extend it by Slodkowski's $\lambda-$lemma to a holomorphic motion of $\hat{\mathbb{C}} \setminus U_\lambda$ over Λ, which we still denote by h_λ.

We define a family of Beltrami forms $(\mu_\lambda)_{\lambda\in\Lambda}$ on $\hat{\mathbb{C}}$ in the following way. For each λ first consider the pullback under h_λ of the standard complex structure on $\hat{\mathbb{C}} \setminus U_\lambda$.

Then extend μ_λ to $\hat{\mathbb{C}} \setminus K(f_\infty)$ by pulling it back successively under f_{λ_∞}, in such a way that it is f_{λ_∞}-invariant on $U_{\lambda_\infty} \setminus K(f_{\lambda_\infty})$; finally extend it to $K(f_{\lambda_\infty})$ by setting $\mu_\lambda = 0$ on $K(f_{\lambda_\infty})$.

Choose any 3 distinct points $z_1, z_2, z_3 \in \overline{V_{\lambda_\infty}} \setminus U_{\lambda_\infty}$. By the Measurable Riemann Mapping Theorem, there exists a holomorphic family of quasiconformal maps $H_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which integrate μ_λ and such that $H_\lambda(z_i) = h_\lambda(z_i)$, $1 \leq i \leq 3$. This induces a family of holomorphic maps $G_\lambda : H_\lambda(U_{\lambda_\infty}) \to H_\lambda(V_{\lambda_\infty})$, defined as

$$G_\lambda := H_\lambda \circ f_{\lambda_\infty} \circ H_\lambda^{-1},$$

which are all holomorphic since $\mu_\lambda = f_{\lambda_\infty}^* \mu_\lambda$ on U_{λ_∞}, and are all hybrid equivalent to f_{λ_∞} since $\mu_\lambda = 0$ a.e. on $K(f_{\lambda_\infty})$. Additionnally, $G_\lambda(z)$ depends holomorphically on λ (see e.g. [BF14, Lemma 1.40]).

On the other hand, recall that for all $n \in \mathbb{N}$, f_{λ_n} and f_{λ_∞} are both hybrid equivalent to T_α by respective hybrid equivalences ψ_n and ψ_∞. Set $\varphi_n := \psi_n \circ \psi_\infty^{-1}$, which is a hybrid equivalence between f_{λ_∞} and f_{λ_n}. By Proposition 6.23, for all n large enough, $h_{\lambda_n} : \hat{\mathbb{C}} \setminus U_{\lambda_\infty} \to \hat{\mathbb{C}} \setminus U_{\lambda_n}$ extends

to a quasiconformal homeomorphism $\widetilde{h}_{\lambda_n} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which coincides with φ_n on $K(f_{\lambda_\infty})$ and such that, for all $z \in U_{\lambda_n}$,

$$f_{\lambda_n}(z) := \widetilde{h}_{\lambda_n} \circ f_{\lambda_\infty} \circ \widetilde{h}_{\lambda_n}^{-1}(z).$$

Hence we have the following commuting diagram.

$$\begin{array}{ccc}
U_{\lambda_n} & \xrightarrow{f_{\lambda_n}} & V_{\lambda_n} \\
\widetilde{h}_{\lambda_n} \big\uparrow & & \big\uparrow \widetilde{h}_{\lambda_n} \\
U_{\lambda_\infty} & \xrightarrow{f_{\lambda_\infty}} & V_{\lambda_\infty} \\
H_{\lambda_n} \big\downarrow & & \big\downarrow H_{\lambda_n} \\
H_{\lambda_n}(U_{\lambda_\infty}) & \xrightarrow{G_{\lambda_n}} & H_{\lambda_n}(V_{\lambda_\infty})
\end{array}$$

We claim that $\widetilde{h}_n = H_{\lambda_n}$. Let $\sigma_n := \frac{\overline{\partial}\widetilde{h}_{\lambda_n}}{\partial\widetilde{h}_{\lambda_n}}$ be the Beltrami form associated to $\widetilde{h}_{\lambda_n}$. Since $\widetilde{h}_{\lambda_n}$ coincides with h_{λ_n} on $\hat{\mathbb{C}} \setminus U_{\lambda_\infty}$, we have $\sigma_n = \mu_{\lambda_n}$ on $V_{\lambda_\infty} \setminus U_{\lambda_\infty}$. Therefore, since both are f_{λ_∞}-invariant, we also have $\sigma_n = \mu_{\lambda_n}$ on $\hat{\mathbb{C}} \setminus K(f_{\lambda_\infty})$; and $\sigma_n = \mu_{\lambda_n} = 0$ a.e. on $K(f_{\lambda_\infty})$. Therefore $\sigma_n = \mu_{\lambda_n}$. Finally, both integrating maps H_{λ_n} and $\widetilde{h}_{\lambda_n}$ must be equal since they coincide on the 3 points z_1, z_2, z_3 introduced above.

Hence $f_{\lambda_n} = H_{\lambda_n} \circ f_{\lambda_\infty} \circ H_{\lambda_n}^{-1}$ for all $n \in \mathbb{N}$. By the Identity Theorem applied to $\lambda \mapsto f_\lambda(z) - H_\lambda \circ f_{\lambda_\infty} \circ H_\lambda^{-1}(z)$, we therefore have $f_\lambda = H_\lambda \circ f_{\lambda_\infty} \circ H_\lambda^{-1}$ for all λ in a neighborhood of λ_∞. But this would imply that $J(f_\lambda)$ is connected for all λ in a neighborhood of λ_∞, contradicting $\lambda_\infty \in \partial\mathcal{T}(\mathbf{f})$. □

6.3.4. *Bijectivity and proof that χ is a homeomorphism.* We start by proving surjectivity, via topological holomorphicity.

Definition 6.25 (Local degree)**.** Let X and Y be two oriented (real) surfaces, and $h : X \to Y$ a continuous map. Let $x \in X$ be such that x is isolated in $h^{-1}(\{h(x)\})$.

Let U and V be simply connected neighborhoods of x and $y := h(x)$ respectively, and such that $h(U) \subset V$ and $\{x\} = U \cap h^{-1}(\{y\})$. If γ is a loop in $U \setminus \{x\}$ with winding number 1 around x then the local degree $i_x(h)$ is the winding number of $h \circ \gamma$ around y, that is

$$i_x(h) = \operatorname{ind}(h \circ \gamma, y).$$

We recall the classical topological argument principle, see e.g. [Lyu24, Proposition 3.9], .

Proposition 6.26 (Topological Argument Principle)**.** *Let $D \subset \mathbb{R}^2$ be a domain bounded by a Jordan curve γ, $h : \overline{D} \to \mathbb{R}^2$ be a continuous map such that h does not take a value y on γ. Assume that the preimage $\{h^{-1}(y)\}$ form a discrete set in D. Then*

$$\operatorname{ind}(h \circ \gamma, y) = \sum_{x \in h^{-1}(y)} i_x(h)$$

Notice that given γ, for any $\widetilde{x}$ sufficiently close to x,

$$\operatorname{ind}(h \circ \gamma, h(x)) = \operatorname{ind}(h \circ \gamma, h(\widetilde{x})) \geq i_{\widetilde{x}}(h). \tag{6.4}$$

Definition 6.27 (Topologically holomorphic)**.** Let X and Y be two oriented (real) surfaces, and $h : X \to Y$ be a continuous map. Let $T \subset Y$ be closed, and let $P := h^{-1}(T)$. We will say that h is topologically holomorphic over T if for all $x \in P$, $h^{-1}(\{h(x)\})$ is discrete and $i_x(h) > 0$.

The follwoing propositions say that in our case it suffices to prove topological holomorphicity to have surjectivity of χ (c.f. [DH85, Prop.19, p. 323] and [DH85, Thm. 4, p. 326].

Proposition 6.28 (Topological holomorphicity implies surjectivity)**.** *Let X, Y, h, P, T be as above, and assume that h is topologically holomorphic over T and that T is connected. Then $h : P \to T$ is surjective, and for all $y \in T$, its topological degree*

$$\delta := \sum_{x_j \in h^{-1}(y)} i_{x_j}(h) \geq 1 \tag{6.5}$$

is independent from y and is called the degree of h.

Proposition 6.29. *The map χ is topologically holomorphic over $\mathcal{T}$, hence surjective.*

Proof. We have already proved that fibers of χ are discrete, so we only need to prove that if $\chi(\lambda) \in \mathcal{T}$, then $i_\lambda(\chi) > 0$. We will distinguish two cases, depending on whether $\chi(\lambda) \in \mathring{\mathcal{T}}$ or $\chi(\lambda) \in \partial\mathcal{T}$.

Assume first that $\chi(\lambda) \in \mathring{\mathcal{T}}$. By Lemma 6.18, $\chi(\partial\mathcal{T}(\mathbf{f})) \subset \partial\mathcal{T}$, therefore $\lambda \in \mathring{\mathcal{T}}(\mathbf{f})$. Then, by Propositions 6.20 and 6.21, the map χ is holomorphic near λ, from which it follows that $i_\lambda(\chi) \geq 1$.

Let us now assume that $\chi(\lambda) \in \partial\mathcal{T}$. Let D be a disk around λ small enough that $D \cap \chi^{-1}(\{\chi(\lambda)\}) = \{\lambda\}$, and let $\gamma := \partial D$.

Since $\lambda \in \partial\mathcal{T}(\mathbf{f})$, by Corollary 6.14 there exists $\widetilde{\lambda} \in \mathring{\mathcal{T}}(\mathbf{f})$ arbitrarily close to λ. By (6.4), we can choose $\widetilde{\lambda}$ close enough to λ so that

$$i_\lambda(\chi) := \operatorname{ind}(\chi \circ \gamma, \chi(\lambda)) = \operatorname{ind}(\chi \circ \gamma, \chi(\widetilde{\lambda})).$$

By the Topological Argument Principle we have

$$\operatorname{ind}(\chi \circ \gamma, \chi(\widetilde{\lambda})) = \sum_{\lambda_j \in D \cap \chi^{-1}(\chi(\widetilde{\lambda}))} i_{\lambda_j}(\chi) \geq i_{\widetilde{\lambda}}(\chi).$$

Since χ is holomorphic near $\widetilde{\lambda}$ (see Proposition 6.20), $i_{\widetilde{\lambda}}(\chi) \geq 1$, so we can conclude that $i_\lambda(\chi) \geq 1$ as desired. This proves that χ is indeed topologically holomorphic; surjectivity follows from Proposition 6.28. ☐

Proposition 6.30 (Injectivity in an exterior neighborhood of $\mathcal{T}(\mathbf{f})$)**.** *The map χ is injective on $X := \{\lambda \in \Lambda : v_\lambda \in V_\lambda \setminus \overline{U_\lambda}\}$.*

Proof. We first claim that $\lambda \mapsto h_\lambda^{-1}(v_\lambda)$ is injective on X. To that end, assume that $z := h_{\lambda_1}^{-1}(v_{\lambda_1}) = h_{\lambda_2}^{-1}(v_{\lambda_2})$, for some $\lambda_1, \lambda_2 \in X$. Then $z \in V_{\lambda_*} \setminus \overline{U_{\lambda_*}}$. Let $t \mapsto z_t$ be a continuous curve joining $z_0 := z$ to $z_1 := u_{\lambda_*}$ in the open annulus $V_{\lambda_*} \setminus U_{\lambda_*}$. Consider the family of maps $F_t : \lambda \mapsto h_\lambda(z_t) - v_\lambda$, defined on $\overline{\Lambda}$. For all $\lambda \in \partial\Lambda$, $v_\lambda \in \partial V_\lambda$, hence $v_\lambda \neq h_\lambda(z_t)$ for all $t \in [0,1]$. Moreover, the curves $(F_t : \partial\Lambda \to \mathbb{C})_{t\in[0,1]}$ are homotopic to each other, hence have the same winding number around 0. Since $h_\lambda(u_{\lambda_*}) = u_\lambda$ (by the equivariance property), $F_1(\lambda) = u_\lambda - v_\lambda$, so this winding number is 1, by the properness assumption. It then follows from the Argument Principle that F_0 has a unique zero in Λ. But since $F_0(\lambda_1) = F_0(\lambda_2) = 0$, we have in fact $\lambda_1 = \lambda_2$. Thus the claim that $\lambda \mapsto h_\lambda^{-1}(v_\lambda)$ is injective on X is proved.

We will now prove that χ is injective on X. Let $\lambda_1, \lambda_2 \in X$ be such that $\chi(\lambda_1) = \chi(\lambda_2) =: \alpha$. Recall that by Corollary 5.2, we have $t_\lambda \circ \varphi_\lambda^{-1}(z) = \Psi_{\chi(\lambda)}(z)$, for all $\lambda \in \Lambda$ and for all $z \in \varphi_\lambda(V_\lambda \setminus \overline{U_\lambda})$. In particular, for all $\lambda \in X$, taking $z = \frac{1}{\chi(\lambda)}$ we obtain:

$$t_\lambda(v_\lambda) = \Psi_{\chi(\lambda)}\left(\frac{1}{\chi(\lambda)}\right).$$

By the definition of the tubing map t_λ (Definition 6.5), we therefore have:

$$h_{\lambda_1}^{-1}(v_{\lambda_1}) = h_{\lambda_2}^{-1}(v_{\lambda_2}) = \Psi_\alpha\left(\frac{1}{\alpha}\right).$$

Therefore, by the above, $\lambda_1 = \lambda_2$. ☐

We are now finally ready to finish the proof of Theorem D.

Proof of Theorem D. We have already proved that $\chi : \Lambda \to \mathbb{C}$ is continuous (Lemmas 6.18, 6.19) and Proposition 6.15. We also proved that $\chi : \mathcal{T}(\mathbf{f}) \to \mathcal{T}$ is surjective (Proposition 6.29). Let us now prove that $\chi : \mathcal{T}(\mathbf{f}) \to \mathcal{T}$ is injective, or equivalently, that the topological degree δ of $\chi : \mathcal{T}(\mathbf{f}) \to \mathcal{T}$ is 1 (See Proposition 6.28). By the Argument Principle and the hypothesis that $\{f_\lambda\}_{\lambda\in\Lambda}$ is proper, there is a unique $\lambda_0 \in \Lambda$ such that $u_\lambda = v_\lambda$; in other words, $\chi^{-1}(\{0\}) = \{\lambda_0\}$. It is then enough to prove that $i_{\lambda_0}(\chi) = 1$ (by Proposition 6.28).

Let $\gamma \subset \Lambda$ be a Jordan curve close enough to $\partial\Lambda$ that $\gamma \subset X$, and that γ winds once around λ_0. Then $i_{\lambda_0}(\chi) = \mathrm{ind}(\chi \circ \gamma, \chi(\lambda_0)) = \mathrm{ind}(\chi \circ \gamma, 0)$. By Proposition 6.30, χ is injective on γ; therefore $\chi \circ \gamma$ is a Jordan curve, and the domain it bounds contains 0. Therefore

$$\delta = i_{\lambda_0}(\chi) = \mathrm{ind}(\chi \circ \gamma, 0) = 1,$$

and $\chi : \mathcal{T}(\mathbf{f}) \to \mathcal{T}$ is indeed injective. Since $\mathcal{T}(\mathbf{f})$ is compact, it follows that $\chi : \mathcal{T}(\mathbf{f}) \to \mathcal{T}$ is a homeomorphism.

Finally, by Propositions 6.20 and 6.21, χ is holomorphic on $\mathring{\mathcal{T}}(\mathbf{f})$. □

7. Existence of proper equipped TL families and proof of Theorem C

Our goal in this section is to show that, for any natural family of meromorphic maps (under some non-exceptionaliity conditions), copies of the Tandelbrot set appear nearby almost any parameter in the activity locus of a given asymptotic value. This is the content of Theorem C. Furthermore, we shall prove Corollary D, stating that in the presence of critical points and asymptotic values, any such open set intersecting the bifurcation locus will contain copies of the Tandelbrot set or of the connectedness locus of $z^d + c$ for some $d \geq 2$.

We begin with the following observation, which we will use repeatedly below. Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of meromorphic maps, and let $\lambda_0 \in M$, v_{λ_0} an isolated asymptotic value be such that for all quasidisk containing v_{λ_0}, f_{λ_0} has a tract above v_{λ_0} which is a quasidisk. We claim that this property automatically extends to all f_λ.

Indeed, since the family is natural, there exists holomorphic motions φ_λ, ψ_λ of $\hat{\mathbb{C}}$ with basepoint λ_0 such that for all $\lambda \in M$,

$$f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}.$$

Now, if D is a quasidisk around v_λ for some $\lambda \in M$, then $\varphi_\lambda^{-1}(D)$ is a quasidisk around v_{λ_0}, so by assumption there is a tract T for f_{λ_0} above $\varphi_\lambda^{-1}(D)$ which is a quasidisk. But then $\psi_\lambda(T)$ is also a quasidisk, and is a tract above D for f_λ.

The next lemma allows us to assume that the starting parameter is a virtual cycle parameter.

Lemma 7.1 (Density of non-critical virtual cycles)**.** *Under the assumptions of Theorem C, there exists λ_0 arbitrarily close to 0 and $m \in \mathbb{N}^*$ such that $f_{\lambda_0}^m(v_{\lambda_0}) = \infty$, and $|(f_{\lambda_0}^m)'(v_{\lambda_0})| \neq 0$ (in the spherical metric).*

Proof. Let p_0 be a simple pole for f_0 which is not a singular value. Let p_λ denote its holomorphic motion as a pole given by the Implicit Function Theorem, near $\lambda = 0$. In view of Proposition 2.23 in [ABF24] (see Proposition 2.20), up to replacing 0 with a nearby parameter λ_1, we can assume that there exists $n \in \mathbb{N}$ such that $f_{\lambda_1}^n(v_{\lambda_1}) = \infty$.

By assumption, v_0 does not persistently have a critical point in its forward orbit; and since critical points form a discrete subset of $\mathbb{C}$ and ψ_λ maps the critical set of f_0 to those of f_λ, the set

$$C_n := \{\lambda \in \mathbb{D} : (f_\lambda^n)'(v_\lambda) = 0\}$$

is discrete near λ_1 (note however that the set C_n is not discrete in $\mathbb{D}$, as it accumulates on parameters λ such that $f_\lambda^k(v_\lambda) = \infty$ for some $k < n$). If $\lambda_1 \notin C_n$, then we are done. Otherwise, there exists $\delta > 0$ such that $\mathbb{D}(\lambda_1, \delta) \cap C_n = \{\lambda_1\}$. By [ABF21, Prop. 2.13] (see the Shooting Lemma, Proposition 2.22), there exists $\lambda_0 \in \mathbb{D}(\lambda_1, \delta) \setminus \{\lambda_1\}$ such that $f_{\lambda_0}^{n+1}(v_{\lambda_0}) = p_{\lambda_0}$; although not stated explicitly, the proof of [ABF21, Prop. 2.13] also implies that $f'_{\lambda_0}(f_{\lambda_0}^n(v_{\lambda_0})) \neq 0$. Finally, by definition, p_{λ_0} is a simple pole, hence not a critical point.

The lemma is therefore proved, with $m := n+2$. □

Under the assumptions of Theorem C, let λ_0 be the virtual cycle parameter given by Lemma 7.1, hence arbitrarily close to $\lambda = 0$ and such that $f_{\lambda_0}^m(v_{\lambda_0}) = \infty$ for some $m \in \mathbb{N}$ which is fixed from now on. By virtue of the parameter version of the Straightening Theorem (Theorem D), to prove Theorem C, it suffices to prove the the existence of proper equipped TL families in a region nearby λ_0. Hence our goal is to prove the following.

Theorem 7.2 (Existence of proper equipped TL families)**.** *Under the assumptions of Theorem C, there exists $\Lambda \Subset \mathbb{D}$ simply connected, $U_\lambda \Subset V_\lambda \subset \hat{\mathbb{C}}$ and $N \in \mathbb{N}^*$ such that $\{f_\lambda^N : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ is a proper equipped TL family, with singular values $f_\lambda^{N-1}(v_\lambda)$.*

Proof. In the following, all distances between points in the Riemann sphere are measured in the spherical metric. In particular, we will denote by $\mathbb{D}(\infty, r)$ the open ball of spherical radius $r > 0$ at ∞ in the Riemann sphere.

Let p_0 be a simple pole for f_0 which is not a singular value. Let p_λ denote its holomorphic motion as a pole given by the Implicit Function Theorem. From now on, let $\lambda_0 \in \mathbb{D}$ be given by Lemma 7.1, hence arbitrarily close to 0 and such that $f_{\lambda_0}^m(v_{\lambda_0}) = \infty$ for some $m \in \mathbb{N}$, with a non-zero derivative. Let us fix $n = m$ once and for all.

For any $r > 0$, let $V_\lambda(r)$ denote the connected component of $f_\lambda^{-1}(\mathbb{D}(\infty, r))$ containing p_λ. Let $r > 0$ and $\eta_0 > 0$ be small enough that there is a holomorphic motion $h_\lambda : V_{\lambda_0}(r) \to V_\lambda(r)$ defined on $\mathbb{D}(\lambda_0, \eta_0)$, such that $f_\lambda \circ h_\lambda(z) = f_{\lambda_0}(z)$ and such that $V_\lambda(r) \cap S(f_\lambda) = \varnothing$ for all $\lambda \in \mathbb{D}(\lambda_0, \eta_0)$ (h_λ is given by the Implicit Function Theorem).

Up to reducing r and η_0 if necessary, assume that for all $\lambda \in \mathbb{D}(\lambda_0, \eta_0)$ there exists a well-defined univalent map $k_\lambda : \mathbb{D}(\infty, r) \to \hat{\mathbb{C}}$ which is the inverse branch of f_λ^n mapping $f_\lambda^n(v_\lambda) \in \mathbb{D}(\infty, r)$ to v_λ. Here, we use the assumption that the orbit of v_0 does not contain other singular values.

Let $\delta, \eta > 0$ be the constants given by Lemma 2.25 applied with $\frac{r}{2}$, i.e.: for all $\lambda \in \mathbb{D}(\lambda_0, \eta)$, a suitable tract $T_\lambda(\delta)$ above $\mathbb{D}(v_\lambda, \delta)$ is compactly contained in $\mathbb{D}(\infty, \frac{r}{2})$. Up to reducing η_0 or η if necessary, let us assume without loss of generality that $\eta_0 = \eta$.

Since the family is natural, we may write $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$, where $f := f_{\lambda_0}$ and $\varphi_\lambda, \psi_\lambda$ are holomorphic motions of the Riemann sphere with base point λ_0. Let $G(\lambda) := \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda)$. By [ABF21, Remark 2.12], there exists a neighborhood W_G of ∞ such that G is a topological branched cover over W_G.

Let $W \subset \mathbb{C}$ be any Jordan domain such that $V_{\lambda_0}(r) \Subset W$ and $W \cap S(f_\lambda) = \varnothing$ for all $\lambda \in \mathbb{D}(\lambda_0, \eta)$ (up to reducing η if necessary). Let W_i be any Jordan domain such that

$$V_{\lambda_0}\left(\frac{r}{2}\right) \Subset W_i \Subset V_{\lambda_0}(r)$$

(for instance, one could take $W_i := V_{\lambda_0}\left(\frac{3r}{4}\right)$). Let $\xi > 0$ be small enough that for all $\lambda \in \mathbb{D}(\lambda_0, \xi)$,

(1) $\varphi_\lambda^{-1} \circ h_\lambda(V_{\lambda_0}(r)) \Subset W$
(2) $W_i \Subset \varphi_\lambda(W)$

(3) $h_\lambda(V_{\lambda_0}\left(\frac{r}{2}\right)) \Subset W_i$.

Because infinity is an essential singularity and there are no critical points there exists a sequence of simply connected domains Ω_k, shrinking to ∞, that map univalently to W under f. In particular, we may fix $\Omega \subset \mathbb{C}$ a simply connected domain such that

(1) $f : \Omega \to W$ is a conformal isomorphism
(2) $\Omega \Subset W_G$, and $\overline{\Omega}$ does not contain the image of any branching point of G
(3) For all $\lambda \in \mathbb{D}(\lambda_0, \xi)$, $\operatorname{diam}(\psi_\lambda(\Omega)) \leq \frac{\delta C}{100}$, where $C := \sup_{\lambda\in\mathbb{D}(\lambda_0,\eta)} |((f_\lambda)^n)'(v_\lambda)|$.
(4) $G^{-1}(\Omega) \Subset \mathbb{D}(\lambda_0, \min(\xi, \eta))$.

Let $\hat{\Lambda}$ denote a connected component of $G^{-1}(\Omega)$. By the choice of Ω, $f \circ G : \overline{\hat{\Lambda}} \to \overline{W}$ is a homeomorphism. In particular, $\hat{\Lambda}$ is a Jordan domain. Now let

$$F(\lambda) := f_\lambda^{n+1}(v_\lambda) = \varphi_\lambda \circ f \circ G(\lambda),$$

and for all $z \in V_{\lambda_0}\left(\frac{r}{2}\right)$, let $\gamma_z(\lambda) := h_\lambda(z)$: both of these maps are holomorphic on $\hat{\Lambda}$. Moreover, by the choice of ξ, we have $\gamma_z(\hat{\Lambda}) \Subset F(\hat{\Lambda})$. Therefore, by the Argument Principle, there exists a unique $\lambda_z \in \hat{\Lambda}$ such that $F(\lambda_z) = \gamma_z(\lambda_z)$. Then, by definition of h_λ, we have

$$f_{\lambda_z}^{n+2}(v_{\lambda_z}) = f_{\lambda_z} \circ F(\lambda_z) = f_{\lambda_z} \circ \gamma_z(\lambda_z) = f_{\lambda_z} \circ h_{\lambda_z}(z) = f_{\lambda_0}(z) \in \mathbb{D}(\infty, \frac{r}{2}).$$

In particular, we have proved that for any $y \in \mathbb{D}(\infty, \frac{r}{2})$, there exists a unique $\lambda \in \hat{\Lambda}$ such that $f_\lambda^{n+2}(v_\lambda) = y$. We let

$$\Lambda = \left\{\lambda \in \hat{\Lambda} : f_\lambda^{n+2}(v_\lambda) \in \mathbb{D}(\infty, \frac{r}{2})\right\}.$$

Since the map $\lambda \mapsto f_\lambda^{n+2}(v_\lambda)$ is a conformal isomorphism from Λ to $\mathbb{D}(\infty, \frac{r}{2})$ by the above, Λ is also a Jordan domain.

Since $f_\lambda^{n+1}(v_\lambda) \in V_\lambda(\frac{r}{2})$ and since $V_\lambda(\frac{r}{2})$ is simply connected and disjoint from $S(f_\lambda)$, the connected component of $f_\lambda^{-(n+1)}(V_\lambda(\frac{r}{2}))$ containing v_λ is simply connected. We will choose an appropriate tract U_λ above it, and prove that for all $\lambda \in \Lambda$, the map

$$g_\lambda := f_\lambda^{n+3} : U_\lambda \to \mathbb{D}(\infty, \frac{r}{2}) \setminus \{f_\lambda^{n+2}(v_\lambda)\}$$

is a tangent-like map. We will then prove that $\{g_\lambda : U_\lambda \to V\}_{\lambda\in\Lambda}$ is a proper equipped family of tangent-like maps with singular values $f_\lambda^{n+2}(v_\lambda)$.

For all $\lambda \in \Lambda$, let $V_\lambda^{(j)}$ denote the connected component of $f_\lambda^{-j}(V_\lambda\left(\frac{r}{2}\right))$ containing $f_\lambda^{n+1-j}(v_\lambda)$. Since $V_\lambda\left(\frac{r}{2}\right) \Subset W$ and $W \cap S(f_\lambda) = \varnothing$ for all $\lambda \in \Lambda$, there is a univalent inverse branch ℓ_λ of f_λ^{-1}, mapping $V_\lambda\left(\frac{r}{2}\right)$ conformally to $V_\lambda^{(1)}$ and depending holomorphically on λ.

We claim that for all $\lambda \in \Lambda$, $V_\lambda^{(1)} \subset \psi_\lambda(\Omega)$. Indeed, recall that for all $\lambda \in \Lambda$, $G(\lambda) = \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda) \in \Omega$; hence $f_\lambda^n(v_\lambda) \in \psi_\lambda(\Omega)$. Moreover, f maps Ω univalently to W, therefore f_λ maps $\psi_\lambda(\Omega)$ univalently to $\varphi_\lambda(W)$. By the choice of ξ, we have $V_\lambda(\frac{r}{2}) \subset \varphi_\lambda(W)$ for all $\lambda \in \mathbb{D}(\lambda_0, \xi)$, hence for all $\lambda \in \Lambda$. Therefore $V_\lambda^{(1)} \subset \psi_\lambda(\Omega)$, as claimed.

By the choice of Ω (condition (3)) we then have

$$\operatorname{diam}(V_\lambda^{(1)}) \leq \operatorname{diam}(\psi_\lambda(\Omega)) \leq \frac{\delta C}{100},$$

for all $\lambda \in \Lambda$. In particular, without loss of generality, $V_\lambda^{(1)} \Subset \mathbb{D}(\infty, r)$, so that $k_\lambda \circ \ell_\lambda : W \to \mathbb{C}$ is a well-defined inverse branch of $f_\lambda^{-(n+1)}$, mapping $V_\lambda\left(\frac{r}{2}\right)$ to $V_\lambda^{(n+1)}$. By the above estimate on $\mathrm{diam}(V_\lambda^{(1)})$, the choice of δ (cf. Lemma 2.25) and Koebe's theorem applied to k_λ, we have $\mathrm{diam}(V_\lambda^{(n+1)}) \leq \delta$ for all $\lambda \in \Lambda$.

Let $w_\lambda := k_\lambda \circ \ell_\lambda \circ h_\lambda \circ f_{\lambda_0}^{n+2} : V_{\lambda_0}^{(n+1)} \to V_\lambda^{(n+1)}$. For all $\lambda \in \Lambda$, the map w_λ is injective, as a composition of univalent maps. Moreover, since k_λ, h_λ and ℓ_λ are all holomorphic in (λ, z), it follows that for all $z \in V_{\lambda_0}^{(n+1)}$, the map $z \mapsto w_\lambda(z)$ is holomorphic. (In fact, more is true: $(\lambda, z) \mapsto w_\lambda(z)$ is holomorphic, but we do not require this). In particular, w_λ is a holomorphic motion of $V_\lambda^{(n+1)}$ over Λ; and by construction, $w_\lambda(v_{\lambda_0}) = v_\lambda$. By assumption, there is a logarithmic tract U_{λ_0} above $V_{n+1}(\lambda_0)$ which is a quasidisk.

Let U_λ denote the corresponding tract above $V_\lambda^{(n+1)}$, given by Lemma 2.24: since U_λ is given by a holomorphic motion of U_{λ_0} which is itself a quasidisk, U_λ is also a quasidisk. By construction, $U_\lambda \Subset \mathbb{D}(\infty, \frac{r}{2}) = V$, and $f_\lambda^{n+3} : U_\lambda \to \mathbb{D}(\infty, \frac{r}{2}) \setminus \{f_\lambda^{n+2}(v_\lambda)\}$ is a universal cover. Therefore $g_\lambda : U_\lambda \to V \setminus \{f_\lambda^{n+2}(v_\lambda)\}$ is indeed a TL map. By construction, Λ is a Jordan domain, and $f_\lambda^{n+2}(v_\lambda) \in V := \mathbb{D}(\infty, \frac{r}{2})$. Additionally, $\lambda \mapsto f_\lambda^{n+2}(v_\lambda)$ is an isomorphism between $\overline{\Lambda}$ and $\overline{\mathbb{D}}(\infty, \frac{r}{2})$, so $\lambda \mapsto f_\lambda^{n+2}(v_\lambda)$ turns once around ∞ as λ turns once along $\partial\Lambda$.

With this, we have proved that $\{f_\lambda^{n+3} : U_\lambda \to V\}_{\lambda \in \Lambda}$ is a TL family with singular values $f_\lambda^{n+2}(v_\lambda)$. Properness follows from the fact that for any $y \in \mathbb{D}(\infty, \frac{r}{2})$, there exists a unique $\lambda \in \hat{\Lambda}$ such that $f_\lambda^{n+2}(v_\lambda) = y$.

It only remains to prove that it is equipped, i.e. that there is a holomorphic motion $\widetilde{m}_\lambda : \overline{V} \setminus U_{\lambda_0} \to V \setminus U_\lambda$ such that for all $\lambda \in \Lambda$, for all $z \in \partial U_{\lambda_0}$,

$$g_\lambda \circ \widetilde{m}_\lambda(z) = \widetilde{m}_\lambda \circ g_{\lambda_*}(z),$$

for some basepoint $\lambda_* \in \Lambda$. By construction, there is a holomorphic motion $m_\lambda : \overline{U_{\lambda_0}} \to \overline{U_\lambda}$ given by Lemma 2.24 such that for all $\lambda \in \Lambda$, for all $z \in \overline{U_{\lambda_0}}$,

$$g_\lambda \circ m_\lambda(z) = g_{\lambda_0}(z).$$

Let $\lambda_* \in \Lambda$ be an arbitrary parameter, and set

$$\widetilde{m}_\lambda(z) = \begin{cases} m_\lambda \circ m_{\lambda_*}^{-1}(z) & \text{if } z \in \partial U_{\lambda_*} \\ z & \text{if } z \in \partial \mathbb{D}(\infty, \frac{r}{2}). \end{cases}$$

This defines an equivariant holomorphic motion of ∂U_λ based at λ_*, i.e. for all $\lambda \in \Lambda$, for all $z \in \partial U_{\lambda_*}$,

$$g_\lambda \circ \widetilde{m}_\lambda(z) = \widetilde{m}_\lambda \circ g_{\lambda_*}(z).$$

Finally, we can extend $\widetilde{m}_\lambda$ to a holomorphic motion of $\hat{\mathbb{C}}$ over Λ by Slodkowski's λ-lemma, which we still denote by $\widetilde{m}_\lambda$. By continuity of $\widetilde{m}_\lambda$ and since $\widetilde{m}_\lambda(\partial U_{\lambda_*}) = \partial U_\lambda$ and $\widetilde{m}_\lambda(\partial V) = \partial V$, we have $\widetilde{m}_\lambda(\overline{V} \setminus U_{\lambda_*}) = \overline{V} \setminus U_\lambda$, as desired. □

This concludes the proof of Theorem C.

The remaining of the section is devoted to the proof of Corollary 1.2. Following [McM00], we recall here the notion of unramified points.

Definition 7.3 (Unramified points)**.** Let f be a transcendendal meromorphic map, and let $z \in \hat{\mathbb{C}}$. We way that $y \in \mathbb{C}$ is an unramified preimage of order n of z if there exists $n \in \mathbb{N}^*$ such that $f^n(y) = z$ and f^n is locally univalent near y. If z has unramified preimages of arbitrarily high order, we say that z is unramified.

The next lemma asserts that if c_λ is an active critical point which is not persistently a Picard exceptional value, then after small perturbations we can find an unramified Misiurewicz critical point. The assumption that c_λ is not persistently a Picard exceptional value is necessary, since for instance if c_λ is always an omitted value, then it can never be unramified. The proof is therefore slightly different than in the case of rational maps. Compare with [McM00, Prop. 2.4 and 2.5].

Lemma 7.4 (Density of Misiurewicz critical points)**.** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a natural family of finite type meromorphic maps, where $\Lambda \subset \mathbb{C}$ is a domain. Let c_λ be a marked critical point, active at $\lambda_1 \in \Lambda$, and assume that there exists $\lambda_0 \in \Lambda$ such that c_{λ_0} is not a Picard exceptional value. Then, there exists some λ_2 arbitrarily close to λ_1 such that f_{λ_2} has an unramified critical point which is Misiurewicz.*

Proof. Let N be the number of singular values. We start with the following observations: if c_λ has at least $N+1$ unramified preimages of order k, then it has infinitely many unramified preimages of order $k+1$. Indeed, there is at least one unramified preimage of order k which is not a singular value, and therefore it itself has infinitely many unramified preimages. In particular, if c_λ has $N+1$ unramified preimages of order 1, then c_λ is unramified.

First, we claim that the set of $\lambda \in \Lambda$ such that c_λ is a Picard exceptional value is discrete. Indeed, let $\mathcal{V}_\lambda$ denote the set of Picard exceptional values of f_λ. Since the family $\{f_\lambda\}_{\lambda\in\Lambda}$ is natural, we may write $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$, with $\varphi_{\lambda_0} = \psi_{\lambda_0} = \mathrm{Id}$, which implies that $\mathcal{V}_\lambda = \varphi_\lambda(\mathcal{V}_{\lambda_0})$. By Picard's Theorem, f_{λ_0} has at most two Picard exceptional values on the sphere, say v_1, v_2. It follows that, if λ is such that c_λ is Picard exceptional, then either $c_\lambda = \varphi_\lambda(v_1)$ or $c_\lambda = \varphi_\lambda(v_2)$. On the other hand, $c_\lambda = \psi_\lambda(c_{\lambda_0})$. Hence the set of $\lambda \in \Lambda$ such that c_λ is a Picard exceptional value satisfy the equation $\varphi_\lambda(v_i) = \psi_\lambda(c_0)$ for either $i = 1$ or $i = 2$. Both equations are holomorphic in λ and are not satisfied for $\lambda = \lambda_0$, hence their set of solutions is discrete in the one-dimensional parameter space Λ.

Hence, by density of Misiurewicz parameters (2.21), there exists λ_2 arbitrarily close to λ_1 such that c_{λ_2} is Misiurewicz and not Picard exceptional. In particular, c_{λ_2} has infinitely many preimages, only finitely many of which can be singular values (since the family is of finite type). We may therefore choose a preimage y_1 of c_λ such that y_1 is not a singular value (but might possibly be a critical point). If y_1 is not a critical point, then it is an unramified preimage of c_{λ_2}, and since it is not a singular value, it itself has infinitely many unramified preimages. Then, by our observation, c_{λ_2} is unramified and we are done. Otherwise, y_1 is a critical point which is not a singular value. But then y_1 has infinitely many unramified preimages of order 1 (since it is not a singular value), hence y_1 is unramified; and since $f_{\lambda_2}(y_1) = c_{\lambda_2}$, y_1 is also Misiurewicz. □

We now recall the result from McMullen below. It is stated for rational maps, however the proof is purely local and holds for transcendental meromorphic maps. The exact statement is also more technical but more precise; we have extracted only what we will use.

Theorem 7.5 ([McM00], Theorem 4.1)**.** *If $\{f_\lambda\}_{\lambda\in\Lambda}$ is a family of rational or meromorphic maps such that f_{λ_*} has a marked unramified critical point c_λ of constant degree $d \geq 2$ which is Misiurewicz, then for any neighborhood W of λ_*, there is a quasiconformal embedding $\chi : \partial\mathcal{M}_d \to \mathrm{Bif} \cap W$.*

Recall that in the case of a natural family, all critical points are always marked and have constant local degree throughout the family. We are now ready to prove Corollary 1.2.

Proof of Corollary 1.2. Let λ_1 be a parameter in the bifurcation locus and let W be an arbitrary neighborhood of λ_1. By characterization of $\mathcal{J}$-stability (Theorem 2.19) there is at least one active singular value v_{λ_1} at λ_1: this active singular value must be either critical or asymptotic.

- If v_{λ_1} is a critical value: then there is an active critical point c_{λ_1}. By Lemma 7.4, there exists $\lambda_2 \in W$ such that f_{λ_2} has an unramified critical point which is Misiurewicz. By Theorem 7.5, there is a topological embedding $\chi : \partial\mathcal{M}_d \to \mathrm{Bif} \cap W$ for some $d \geq 2$.
- If v_{λ_1} is an asymptotic value: if its forward orbit does not persistently contain a critical point, then we can apply directly Theorem C: in that case, we have a topological embedding $\chi : \partial\mathcal{T} \to \mathrm{Bif} \cap W$. It its forward orbit does contain a critical point, then that critical point is active, and we are back to the previous case.

□

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