BIFURCATIONS FOR FAMILIES OF AHLFORS ISLAND MAPS

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ABSTRACT. We extend Mañé-Sad-Sullivan and Lyubich's equivalent characterization of stability to the setting of Ahlfors island maps, which include notably all meromorphic maps. As a consequence we also obtain the density of J-stability for finite type maps in the sense of Epstein.

1. INTRODUCTION

A fundamental type of questions in complex dynamics relates to parameter spaces. Consider a complex connected manifold M and a holomorphic family $\{f_\lambda\}_{\lambda\in M}$ of holomorphic self-maps $f_\lambda : X \to X$ of a Riemann surface X, or more generally, of partially defined maps $f_\lambda : W_\lambda \to X$, with $W_\lambda \subset X$. How are global dynamics of f_λ affected by a variation of the parameter λ? Of particular interest is the set of parameters $\lambda \in M$ for which the global dynamics is stable under perturbation of the parameter in some sense; this is called the *stability* locus, and its complement is the *bifurcation* locus. For families of rational maps on $\mathbb{P}^1$ of given degree, much is understood. In the seminal works [MSS83] and [Lyu84], it has been proven that several possible notions of stability coincide: continuous motion of the Julia set (J-stability), stability of periodic orbits, and stability of critical orbits (passivity). As a consequence, the authors obtained the following crucial result: stability is open and dense in any holomorphic family of rational maps.

In [EL92], Eremenko and Lyubich extended this result to the setting of *natural families* of *finite type entire maps*. An entire map is of finite type if it has only finitely many singular values (see Section 2.1 for the definition of singular values, and see Definition 1.3 for the definition of a natural family). The key point in the proof is the absence of collisions between periodic cycles and the essential singularity ∞. In [ABF21], the analogous result was proven for natural families of finite type *meromorphic* maps. In this setting, due to the presence of poles, it turns out that collisions between periodic points and ∞ do actually occur, creating a new type of bifurcation to be investigated.

In this paper, we extend the equivalent characterizations of stability to a much broader class of maps, called Ahlfors Island maps, and we show that stability is dense for a subclass of these maps, namely finite type maps. Both of these classes were introduced by Epstein in his PhD thesis ([Eps93]), see also [RR12] and [MR12].

Date: October 5, 2024.

2020 *Mathematics Subject Classification.* Primary 37F46, 30D05, 37F10, 30D30, 37F44.

† Partially supported by the French ANR grant PADAWAN /ANR- 21-CE40-0012-01.

* Partially supported by the French Italian University and Campus France through the Galileo program, under the project From rational to transcendental: complex dynamics and parameter spaces; by GNAMPA, INdAM; by PRIN Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics n.2017JZ2SW5.

‡ Partially supported by grants PID2023-147252NB-I00 and CEX2020-001084-M (Maria de Maeztu Excellence program) from the Spanish state research agency, and ICREA Acadèmia 2020 from the Catalan government.

Definition 1.1 (Ahlfors island map). *Let X be a compact Riemann surface and $W \subset X$ a non-empty open set. Let $f : W \to X$ be a holomorphic map. We say that f has the N-island property if given any N Jordan domains $D_1, \dots, D_N \subset X$ whose closures are pairwise disjoint and any open set U intersecting ∂W, there exists $1 \leq i \leq N$ and an open set $\Omega \Subset U \cap W$ such that $f : \Omega \to D_i$ is a conformal isomorphism.*

If there exists $N \geq 1$ such that f has the N- island property, we say that f is an Ahlfors island map.

Definition 1.2 (Finite type map). *Let X be a compact Riemann surface, and let $W \subset X$ be a non-empty open set. Let $f : W \to X$ be a holomorphic map. We say that f is a finite type map on X if*

(1) f is non-constant on every connected component of W
(2) f has no removable singularities
(3) The set of singular values $S(f)$ is finite.

Although it is not immediately apparent from these definitions, finite type maps form a subclass of Ahlfors island maps: Epstein proved that finite type maps have the $N+1$-island property, where $N = \operatorname{card} S(f)$ (see [Eps93, Proposition 9 p. 88]). The following are examples of finite type maps:

- Rational maps on the Riemann sphere $\mathbb{P}^1$ of degree $d \geq 2$.
- Entire or meromorphic maps $f : \mathbb{C} \to \mathbb{P}^1$ with finitely many singular values.
- Maps $f : \mathbb{P}^1 \setminus E \to \mathbb{P}^1$ meromorphic outside of a compact totally disconneted set E (see [BDH04, BDH01]) and with finitely many singular values.
- Universal covers $f : \mathbb{D} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ of the thrice punctured sphere, with $X = \mathbb{P}^1$, $W = \mathbb{D}$; in that case $S(f) = \{0, 1, \infty\}$.
- Horn maps of rational maps (see [Eps93] for a proof of this fact, and see e.g. [BEE13] for definitions and properties of horn maps). These maps arise as geometrical limits of rational maps with a parabolic fixed point.
- Any composition or iterate of finite type maps.

More generally, examples of Ahlfors island maps include:

- Arbitrary meromorphic maps: by a deep theorem of Ahlfors (see [Ahl35], [Ber00, Theorem A.2]), any transcendental meromorphic map $f : \mathbb{C} \to \mathbb{P}^1$ has the 5-island property, hence is an Ahlfors island map.
- Horn maps of semi-parabolic Hénon maps (introduced in [BSU17]), satisfy a property which is very close to the Ahlfors island property, called the small island property (see [AB24]). All of our results on Ahlfors island maps also apply to maps with the small island property, although we have chosen to work within the more established setting of Ahlfors island maps.
- Any composition or iterate of Ahlfors island maps is also an Ahlfors island map.

Observe that if f has the N-island property, then f can omit at most $N-1$ points in X. So for instance if $C \subset \mathbb{P}^1$ is a closed infinite set, and $f : \mathbb{D} \to \mathbb{P}^1 \setminus C$ is a covering map, then f is not an Ahlfors island map.

When working with families of rational maps, it is natural to ask that the degree remains fixed throughout the family. We require a similar type of condition for the families of dynamical systems that we will consider, although they are of course typically of infinite degree. A somewhat stronger but very convenient notion is that of so-called natural families, to our knowledge first introduced in [EL92] in the context of entire maps.

Definition 1.3 (Natural families). *Let $f : W \to X$ be a holomorphic map, where X is a compact Riemann surface, $W \subset X$ is an open set, and let M be a connected complex manifold. A natural family is a family of holomorphic maps $\{f_\lambda : W_\lambda \to X\}_{\lambda \in M}$ of the form*

$$f_\lambda := \varphi_\lambda \circ f \circ \psi_\lambda^{-1}, \tag{1}$$

where $\varphi_\lambda, \psi_\lambda : X \to X$ are quasiconformal homeomorphisms depending holomorphically on $\lambda \in M$, and $W_\lambda = \psi_\lambda(W)$.

Let $f_\lambda := \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$ be as above. The following basic but important facts follow directly from the fact that $\varphi_\lambda, \psi_\lambda$ are homeomorphisms:

(a) φ_λ maps $S(f)$ to $S(f_\lambda)$
(b) ψ_λ maps critical points of f to critical points of f_λ, preserving multiplicities
(c) if $f : W \to X$ is a finite type map (respectively an Ahlfors island map), then so are the maps $f_\lambda : W_\lambda \to X$.

Generalizing [EL92] and [GK86], Epstein proved in [Eps93] that one can always embed any finite type map in a (locally) natural family of dimension $\operatorname{card} S(f)$. Moreover, this family satisfies a universal property in the sense that any other natural family can be lifted to it. More recently, these results were partially extended to the case of arbitrary entire maps by Ferreira and van Strien ([FvS23]). We will however not require any of these results, and we refer the reader to [Eps93], [Eps09], and [FvS23].

Finally, we recall the notion of activity or passivity of a singular value (see [Lev81], [McM94]).

Definition 1.4 (Passive singular value). *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of finite type maps. Let $v(\lambda) := \varphi_\lambda(v)$ be a singular value of f_λ depending holomorphically on λ near some $\lambda_0 \in M$. We say that $v(\lambda)$ is* passive *at λ_0 if there exists a neighborhood V of λ_0 in M such that:*

(1) either $f_\lambda^n(v(\lambda)) \in X \setminus W_\lambda$ for some $n \in \mathbb{N}$ and for all $\lambda \in V$; or
(2) the family $\{\lambda \mapsto f_\lambda^n(v(\lambda))\}_{n \in \mathbb{N}}$ is well-defined and normal on V.

We say that $v(\lambda)$ is active *at λ_0 if it is not passive.*

Following classical terminology, we will say that a natural family of Ahlfors island map is J-stable if there is a holomorphic motion of the Julia set respecting the dynamics (see Definition 3.1). The main results in this article are the following.

Theorem 1.5 ($J-$stability of Ahlfors Islands maps). *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of Ahlfors island maps which are not automorphisms of X. Let $\lambda_0 \in M$. Then the following are equivalent:*

(1) there exists a neighborhood of λ_0 on which all singular values are passive.
(2) the family is J-stable on a neighborhood of λ_0.

As in [ABF21], one of the main difficulties in the proof of this type of result is the possibility of collisions between periodic cycles and the boundary ∂W_λ of the domain of definition W_λ of the map f_λ. In [ABF21], a delicate analysis allowed us to relate this phenomenon to the activity of asymptotic values (see in particular Theorem A). However, this analysis used in a crucial way the fact that $\partial W_\lambda = \{\infty\}$ in the case of a family of meromorphic maps. In the more general setting of Ahlfors island maps or even finite type maps, ∂W_λ may contain a continuum, and tracts above asymptotic values may accumulate on such continua in a complicated way. This is why instead of working with the motion of periodic cycles, we consider the motion of the backward orbit of some point $z \in J(f_{\lambda_0})$, and relate this holomorphic motion (or lack thereof) to the activity or passivity of singular values (see Proposition 3.3).

In view of Theorem 1.5, we hereafter define the *stability locus* of a natural family $\{f_\lambda\}_{\lambda\in M}$ as the set of all $\lambda \in M$ for which one of the above equivalent conditions is satisfied, and the *bifurcation locus* as its complement, extending these classical notions to the very general setting of Ahlfors island maps.

In the case of finite type maps, we can obtain one more equivalent characterization of stability.

Theorem 1.6 ($J-$stability of finite type maps). *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type maps which are not automorphisms of X, and let $\lambda_0 \in M$. The following are equivalent:*

(1) *there exists a neighborhood of λ_0 on which all singular values are passive*
(2) *the family is J-stable on a neighborhood of λ_0*
(3) *there is a constant $N \in \mathbb{N}$ and a neighborhood $U \subset M$ of λ_0 such that for all $\lambda \in U$, the period of any attracting cycle is at most N.*

As a consequence, we obtain (as in the case of rational or finite type meromorphic maps):

Corollary 1.7 (Density of $J-$stable maps). *The stability locus is open and dense in natural families of finite type maps.*

On the other hand, using Theorem 1.5 is is very easy to construct families of Ahlfors island maps with robust bifurcations, i.e. for which the set of J-stability is not dense. For instance:

Corollary 1.8 (Bifurcation locus with nonempty interior). *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of Ahlfors island maps and let $\lambda_0 \in M$. Assume that*

(1) *v_{λ_0} belongs to the interior of $S(f_{\lambda_0})$*
(2) *and v_{λ_0} is active at λ_0.*

Then λ_0 is in the closure of the interior of the bifurcation locus.

There are many examples of Ahlfors island maps f_{λ_0} satisfying the first condition of Corollary 1.8; in fact, there exists entire or meromorphic maps f with $S(f) = \mathbb{P}^1$.

Acknowledgements. We wish to thank Adam Epstein and Lasse Rempe for helpful discussions.

2. Activity Locus of Ahlfors Island Maps

2.1. **Preliminaries: Ahlfors island maps, finite type maps and singular values.** Choose an arbitrary hermitian metric on X (the choice is not important since X is compact), and denote by d_X the distance it induces. Unless otherwise stated, distances in X will be measured in the sense of d_X.

We start by recalling the definitions of critical, asymptotic and singular values. Let W, X be two Riemann surfaces and let $f : W \to X$ holomorphic. A point $c \in W$ is *critical* if $f'(c) = 0$. A value $v \in X$ is *critical* if it is the image of a critical point; it is *asymptotic* if there is a curve $\gamma : \mathbb{R}_+ \to W$ such that, as $t \to \infty$, $\gamma(t) \to \partial W$ and $f(\gamma(t)) \to v$. (Notice that if $W \subset X$, we do not require $\gamma(t)$ to converge in X). A *logarithmic tract* over an asymptotic value v is a simply connected domain $T \subset W$ such that $f : T \to D \setminus \{v\}$ is an infinite degree unbranched covering over a topological disk D punctured at v. Any limit point of a (possibly constant) sequence of critical or asymptotic values is called a *singular value*. If we denote by $S(f)$ the set of all singular values of a holomorphic map $f : W \to X$, it is a classical result that $f : W \setminus f^{-1}(S(f)) \to X \setminus S(f)$ is a covering map (assuming that $X \setminus S(f)$ is not empty).

As a consequence, any asymptotic value that is isolated in $S(f)$ admits logarithmic tracts. This is in particular the case for finite type maps, although not in general for Ahlfors island maps.

2.2. **Ahlfors island maps.** We now move on to basic properties of Ahlfors island maps, including a classification of so-called exceptional Ahlfors island maps which will sometimes require separate arguments. Roughly speaking, these exceptional cases correspond to those for which few or no points can escape the domain of definition, and include notably rational and entire maps.

By considering a sequence of neighborhoods shrinking to a boundary point z, the island property implies that if $\partial W \neq \varnothing$, then there are at most finitely many values which do not have infinitely many preimages in W under f, and every such value is a singular value. In particular, if f is an Ahlfors island map of finite degree, then we must have $W = X$.

Definition 2.1 (Fatou and Julia sets). *Let $f : W \to X$ be an Ahlfors island map. The Fatou set $F(f)$ of f is defined as the union of all open subsets $U \subset W$ such that*

(1) either there exists $n \in \mathbb{N}^*$ such that $f^n(U) \cap W = \varnothing$, or
(2) $f^n(U) \subset W$ for all $n \in \mathbb{N}$ and $\{f^n : U \to X\}_{n\in\mathbb{N}}$ is normal.

The Julia set is $J(f) := X \setminus F(f)$.

Observe that by this definition, we have $\partial W \subset J(f)$, where ∂W denotes the boundary of W as a subset of X. Hence if $W \neq X$, then the Julia set is non-empty. The theorem below (stated for finite type maps, but whose proof also works for Ahlfors island map) gives a much stronger statement:

Theorem 2.2 ([Eps93, p. 100]). *Let $f : W \to X$ be an Ahlfors island map which is not an automorphism of X. Then $J(f)$ is non-empty, perfect, and repelling cycles are dense in $J(f)$.*

Definition 2.3 (Exceptional Ahlfors islands maps). *Let $f : W \to X$ be an Ahlfors island map. Let $W_\infty := \operatorname{int} \bigcap_{n\geq 0} f^{-n}(W)$. We say that f is exceptional if either f is an automorphism of X or W_∞ is non-empty and non-hyperbolic.*

Lemma 2.4 (Exceptional Ahlfors islands maps). *An Ahlfors island map $f : W \to X$ is exceptional if and only if it is analytically conjugated to one of the following types:*

(1) a rational self-map of $\mathbb{P}^1$
(2) an affine endomorphism of a complex torus
(3) an entire map
(4) a meromorphic map with exactly one pole, which is also an omitted value
(5) a self-map of $\mathbb{C}^*$ with essential singularities at 0 and ∞
(6) an automorphism of X.

Proof. By definition if f is exceptional and not an automorphism then there is at least one connected component of W_∞ which is not hyperbolic. In particular, there is one connected component of W which is not hyperbolic. Since $W \subset X$, X is also not hyperbolic. This means that W and X are isomorphic to one of the following: $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^*$, or a complex torus.

If $W \simeq \mathbb{P}^1$, then $X = \mathbb{P}^1$ and f is a rational map.

If W is a complex torus, then $W = X$ and f must be an affine endomorphism of X (since endomorphisms of complex tori are all affine).

If $W \simeq \mathbb{C}$, then $X = \mathbb{P}^1$ (since X is compact) and f is a transcendental meromorphic map (since the island property implies infinite degree). Moreover, we must have

$$\operatorname{card} \bigcup_{n\geq 0} f^{-n}(\{\infty\}) \leq 2,$$

since otherwise W would be hyperbolic. This implies that f has at most one pole, and that this pole must be an omitted value; otherwise, by Picard's theorem, we would have $\operatorname{card} \bigcup_{n\geq 0} f^{-n}(\{\infty\}) = \infty$.

Finally, if $W \simeq \mathbb{C}^*$, then the island property implies that f has two essential singularities at 0 and ∞, and as before we must have $\operatorname{card} \bigcup_{n\geq 0} f^{-n}(\{0,\infty\}) \leq 2$, so both 0 and ∞ are omitted values and f is a self-map of $\mathbb{C}^*$.

Conversely, it is clear that maps of the form (1)-(6) are exceptional. □

The following is an immediate consequence of the exhaustive list given above.

Corollary 2.5. *If W is hyperbolic and non-compact, then f is not exceptional.*

Note that the converse is not true, since a meromorphic map with infinitely many poles is not exceptional.

In natural families, either the whole family is exceptional or exceptional maps form a proper analytic subset.

Proposition 2.6 (Natural family with exceptional maps)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of Ahlfors island maps. Then either all maps in M are exceptional, or the set of exceptional maps in M forms a (possibly empty) proper analytic subset of M.*

Proof. Let $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$ and assume that $f := f_{\lambda_0}$ is exceptional. As usual, we may assume that $\varphi_{\lambda_0} = \psi_{\lambda_0} = \mathrm{id}$. If f is an automorphism, so is f_λ for all $\lambda \in M$ and we are done. Otherwise, by Lemma 2.4, either $X = W$ is a complex torus, or $X = \mathbb{P}^1$ and $W = \mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{C}^*$. If $W = \mathbb{P}^1$ or if W is a complex torus, then clearly all maps in the family are exceptional, so we can further reduce to the case where $W = \mathbb{C}$ or $\mathbb{C}^*$, i.e. f is either meromorphic or defined on $\mathbb{C}^*$ with two essential singularities.

The meromorphic case was treated in [ABF21, Prop. 5.4], so we only need to deal with the case where $W = \mathbb{C}^*$, which is similar. By the classification of Lemma 2.4, f omits 0 and ∞. Without loss of generality, we may normalize ψ_λ so that it always fixes 0 and ∞. Then for all $\lambda \in M$, $\varphi_\lambda(0)$ and $\varphi_\lambda(\infty)$ are omitted values of f_λ. By Picard's theorem, f_λ is then exceptional if and only if $\varphi_\lambda(\{0,\infty\}) = \{0,\infty\}$. Indeed, if this relation is not satisfied, then at least one of 0 or ∞ is not a Picard exceptional value, so that say $\operatorname{card} f_\lambda^{-1}(\{0,\infty\}) = \infty$ and f_λ is not exceptional. By connectivity of M, the set of $\lambda \in M$ such that f_λ is exceptional is then exactly

$$E := \{\lambda \in M : \varphi_\lambda(0) = 0 \text{ and } \varphi_\lambda(\infty) = \infty\}.$$

This set is either all of M or a (possibly empty) proper analytic subset of M. □

We record here the following well-known version of Montel's theorem (see e.g. [Mil06, Corollary 3.3]).

Lemma 2.7 (Montel's Theorem)**.** *Let U, V be hyperbolic Riemann surfaces. Then the family of maps $\{f \mid f : U \to V$ holomorphic$\}$ is a normal family.*

Lemma 2.8 below generalizes to the case of non-exceptional Ahlfors island maps the well-known characterization of the Julia set as the closure of the set of prepoles of f. Recall that a transcendental meromorphic map f is non-exceptional if and only if there is at least one non-omitted pole.

Lemma 2.8 (Characterization of $J(f)$)**.** *Let $f : W \to X$ be a non-exceptional Ahlfors island map. Then $J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\partial W)}$.*

Proof. The inclusion $\overline{\bigcup_{n\geq 0} f^{-n}(\partial W)} \subset J(f)$ is always true by definition. Conversely, if $z \notin \overline{\bigcup_{n\geq 0} f^{-n}(\partial W)}$, then there exists a neighborhood U of z such that $f^n(U) \cap \partial W = \varnothing$ for all $n \in \mathbb{N}$. Therefore either there exists $n \in \mathbb{N}$ such that $f^n(U) \cap W = \varnothing$, and hence $z \in F(f)$; or $f^n(U) \subset W$ for all $n \in \mathbb{N}$, i.e. $U \subset W_\infty$. Since f is non-exceptional, W_∞ is hyperbolic. Thus $(f^n|_{W_\infty})_{n\in\mathbb{N}}$ is normal by Lemma 2.7 and $z \in F(f)$. □

For a holomorphic map f with an isolated essential singularity, a point z_0 is called Picard exceptional if and only if $f^{-1}(z_0)$ has finite cardinality. By analogy, with introduce the following terminology.

Definition 2.9 (Picard exceptional value)**.** *Let $f : W \to X$ be an Ahlfors island map with $\partial W \neq \varnothing$. We will say that $v \in X$ is a Picard exceptional value if $f^{-1}(v)$ is finite (possibly empty).*

By definition, if f has the N island property and $\partial W \neq \varnothing$, then it has at most $N-1$ Picard exceptional values. Lemma 2.10 below generalizes the characterization of the Julia set as the closure of the backward orbit of any point z_0 which is not Picard exceptional. One can show that Picard exceptional values are always asymptotic values.

Lemma 2.10 (Preimages of non exceptional values are dense in $J(f)$)**.** *Let $f : W \to X$ be a non-exceptional Ahlfors island map, and let $p \in \mathbb{N}^*$. Let $z_0 \in J(f)$ and assume that z_0 is not a Picard exceptional value. Then $\bigcup_{n\geq 0} f^{-np}(\{z_0\})$ is dense in $J(f)$.*

Proof. Since repelling cycles are dense in the Julia set by Theorem 2.2 we have that $J(f) = J(f^p)$, hence we may assume without loss of generality that $p = 1$.

Let $U \subset W$ be an open set which intersects $J(f)$.

By Lemma 2.8, there exists $n \in \mathbb{N}^*$ and $z \in U$ such that $f^n(z) \in \partial W$.

Since z_0 is not a Picard exceptional value and f is non-exceptional (hence has infinite degree), z_0 has infinitely many preimages : choose $N+1$ such preimages $x_1, \ldots, x_{N+1}$, where f has the N-island property, and let $D_1, \ldots, D_{N+1}$ be Jordan domains with pairwise disjoint closures containing each x_i. By the island property, there exists $1 \leq i \leq N+1$ and $\Omega \Subset f^n(U) \cap W$ such that $f : \Omega \to D_i$ is a conformal isomorphism. In particular, $f^{n+2} : U \cap f^{-n}(\Omega) \to f(D_i)$ is well-defined and surjective; in other words, there exists $z_1 \in U$ such that $f^{n+2}(z_1) = z_0$, as desired. □

2.3. **Activity of singular values and preliminary results.** In this section we collect some results about activity and passivity of singular values in natural families of Ahlfors island maps. In what follows, we fix a natural family $\{f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}\}_{\lambda\in M}$ of Ahlfors island maps.

Given a singular value v_{λ_0} we consider the holomorphic function $v(\lambda) := \varphi_\lambda(v_{\lambda_0})$. Since $\{f_\lambda\}_{\lambda\in M}$ is a natural family, v_λ is a singular value for f_λ for each $\lambda \in M$, of the same type as v_{λ_0}. With this in mind we often refer to $v(\lambda)$ as a singular value, although technically it is a holomorphic function whose value $v(\lambda)$ is a singular value for f_λ for each fixed λ. We also use the equivalent notation $v_\lambda = v(\lambda)$.

Recall the definition of activity/passivity given in the introduction.

Definition 2.11 (Passive singular value). *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type maps. Let $v(\lambda)$ be a singular value (or a critical point) of f_λ depending holomorphically on λ near some $\lambda_0 \in M$. We say that $v(\lambda)$ is* passive *at λ_0 if there exists a neighborhood V of λ_0 in M such that:*

(1) either $f_\lambda^n(v(\lambda)) \in X \setminus W_\lambda$ for some $n \in \mathbb{N}$ and for all $\lambda \in V$; or

(2) the family $\{\lambda \mapsto f_\lambda^n(v(\lambda))\}_{n\in\mathbb{N}}$ is well-defined and normal on V.

We say that $v(\lambda)$ is active *at λ_0 if it is not passive.*

Definition 2.12 (Activity locus). *Given a singular value v_λ we define its* activity locus *as the set of parameters*

$$\mathcal{A}(v_\lambda) = \{\lambda_0 \in M \mid v_\lambda \textit{ is active at } \lambda_0\}.$$

It is important to remark that the concept of activity must be associated only to non-persistent behaviour.

Definition 2.13 (Persistence). *We say that $f_{\lambda_0}^n(v(\lambda_0)) \in X \setminus W_{\lambda_0}$ (resp. $f_{\lambda_0}^n(v(\lambda_0)) \in \partial W_{\lambda_0}$)* persistently *if for all λ in a neighborhood of λ_0, we have $f_\lambda^n(v(\lambda)) \in X \setminus W_\lambda$ (resp. $f_\lambda^n(v(\lambda)) \in \partial W_\lambda$).*

Lemma 2.14 (Persistence property). *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type maps. Let $v(\lambda)$ be a singular value (or a critical point) of f_λ depending holomorphically on $\lambda \in M$ If v_λ. If $n \in \mathbb{N}$ is such that $f_\lambda^n(v_\lambda) \in \partial W_\lambda$ for all λ in an open subset of M, then $f_\lambda^n(v_\lambda) \in \partial W_\lambda$ persistently on M.*

The proof follows almost inmediately from the following lemma, which will also be useful later on.

Lemma 2.15 ([ABF21]). *Let $(\psi_\lambda : X \to X)_{\lambda\in M}$ be a holomorphic family of quasiconformal homeomorphisms, such that $\psi_{\lambda_0} = \mathrm{id}$ and $\dim M = 1$. Let $g : M \to X$ be a holomorphic map and $G(\lambda) := \psi_\lambda^{-1} \circ g(\lambda)$. Then either G is constant, or there are neighborhoods U of λ_0 in M and V of $G(\lambda_0)$ in X such that $G : U \to V$ is a branched cover, ramified only possibly at $G(\lambda_0)$.*

In fact, one could prove that G is even quasiregular ([Ber13]), although we will not require this.

Proof of Lemma 2.14. Let $G(\lambda) := \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda)$. By the previous Lemma, the map $G : M \to X$ is either locally constant (hence constant since M is connected) or open. Since $G(\lambda) \in \partial W$ if and only if $f_\lambda^n(v_\lambda) \in \partial W_\lambda$, and ∂W has empty interior, the map G cannot be open, and is therefore constant. □

The next lemma, though technical, is standard in rational dynamics. It will be used to locally follow holomorphically preimages of a marked point, up to passing to a finite branched cover in parameter space. Since we work with non-rational maps, we give here a detailed proof.

Lemma 2.16 (Holomorphic dependence of preimages). *Let S, X be Riemann surfaces, $U \subset X$ be a domain, and $F : S \times U \to X$ be a holomorphic map such that for all $\lambda \in S$, the map $F(\lambda, \cdot)$ is non-constant on U. Let $\gamma : S \to X$ be a holomorphic map, and let $(\lambda_0, z_i) \in S \times U$ (where $z_i, 1 \le i \le N$ are N distinct points in U) be such that $F(\lambda_0, z_i) = \gamma(\lambda_0)$. Then there* *is a neighborhood V of λ_0 in S, a finite branched cover $\pi : \mathbb{D} \to V$ and holomorphic maps $\mathbb{D} \ni t \mapsto x_i(t) \in U$ such that $x_i(0) = z_i$ and for all $t \in \mathbb{D}$,*

$$F(\pi(t), x_i(t)) = \gamma \circ \pi(t).$$

Proof. Up to restricting U, we may assume without loss of generality that $F(\lambda_0, \cdot)$ extends holomorphically in a neighborhood of $\overline{U}$ in X, and that for all $z \in \partial U$, $F(\lambda_0, z) \neq \gamma(\lambda_0)$.

Let

$$Z := \{(\lambda, (y_1, \dots, y_N)) \in S \times U^N : F(\lambda, y_i) - \gamma(\lambda) = 0 \text{ for } 1 \leq i \leq N\}$$

and let Z_0 denote the irreducible component of Z containing $(\lambda_0, z_1, \dots, z_N)$. Let $\pi_S : (\lambda, x_1, \dots, x_N) \mapsto \lambda$ denote the projection from Z_0 to S. Then Z_0 is an analytic subset of $S \times U^N$ of complex dimension one. Indeed, if Z_0 had higher dimension, then $\pi_S^{-1}(\{\lambda_0\}) \cap Z_0$ would have positive dimension, which would contradict the assumption that $F(\lambda_0, \cdot)$ is non-constant on U.

The set of singular points of Z_0 is discrete (since it is a codimension at least 1 analytic subset of Z_0), and so is the set of critical points of the projection π_S restricted to Z_0. Therefore, there exists a small disk $V \subset S$ containing λ_0 such that $Z_1^* := Z_0 \cap \pi_S^{-1}(V \setminus \{\lambda_0\})$ is smooth and $\pi_S : Z_1^* \to V \setminus \{\lambda_0\}$ has no critical points. Up to taking a smaller V, we may also assume that for all $(\lambda, z) \in V \times \partial U$, $F(\lambda, z) \neq \gamma(\lambda)$. Then the map $\pi_S : Z_1^* \to V \setminus \{\lambda_0\}$ is proper. Indeed, let $K \subset V \setminus \{\lambda_0\}$ denote a compact set, and let $(\lambda_n, z_n) \in \pi^{-1}(K)$. Up to extracting a subsequence, we may assume that $(\lambda_n, z_n) \to (\lambda, z) \in (K \times \overline{U}) \cap Z_1^*$. But by our restriction on V, we have $z \in U$. This proves that $\pi_S^{-1}(K)$ is compact in Z_1^*, hence that $\pi_S : Z_1^* \to V \setminus \{\lambda_0\}$ is proper. It is also surjective (because it is open and closed, and S is connected).

Since $\pi_S : Z_1^* \to V \setminus \{\lambda_0\}$ is proper, surjective and has no critical points, it is a covering map. Therefore, there exists a conformal isomorphism $h : \mathbb{D}^* \to Z_1^*$ and a degree $d \geq 1$ covering map $\pi : \mathbb{D}^* \to V \setminus \{\lambda_0\}$ such that $\pi_S \circ h = \pi$. The map h extends to a holomorphic map $h : \mathbb{D} \to S \times U^N$ such that $h(0) = (\lambda_0, z_1, \dots, z_N)$ and the map π extends to a holomorphic map $\pi : \mathbb{D} \to V$ with $\pi(0) = \lambda_0$. For $1 \leq i \leq N$, let $\pi_i : S \times U^N \to U$ be the projection defined by $\pi_i(\lambda, x_1, \dots, x_N) = x_i$.

We can then set $x_i := \pi_i \circ h$; it is straightforward to check that they have the desired properties. $\square$

We now show that if a singular value is mapped to ∂W_{λ_0} at a parameter $\lambda_0 \in M$, the latter parameter can be perturbed in such a way that the singular value has any prescribed behavior.

Proposition 2.17 (Shooting Lemma). *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of Ahlfors island maps. Let $\lambda_0 \in M$ and $n \geq 0$ be such that a singular value v_λ satisfies $f_{\lambda_0}^n(v_{\lambda_0}) \in \partial W_{\lambda_0}$ non persistently. Let $\lambda \mapsto \gamma(\lambda)$ be a holomorphic map such that $\gamma(\lambda_0)$ is not Picard exceptional. Then we can find λ' arbitrarily close to λ_0 such that*

$$f_{\lambda'}^{n+2}(v_{\lambda'}) = \gamma(\lambda').$$

Since Picard exceptional values are also asymptotic values, Proposition 2.17 applies in particular whenever $\gamma(\lambda_0) \notin S(f_{\lambda_0})$. To prove Proposition 2.17 we need the following lemma.

Lemma 2.18 ([ABF21]). *Let V be a Jordan domain, and let f, g be holomorphic functions in a neighborhood of $\overline{V}$. Suppose that $g(\overline{V}) \subset f(V)$ and $g(\partial V) \cap f(\partial V) = \varnothing$. Then there exists $\lambda \in V$ such that $f(\lambda) = g(\lambda)$.*

Proof of Proposition 2.17. First, we pick an arbitrary one-dimensional slice containing λ_0 in the parameter space M on which $\lambda \mapsto f_\lambda^n(v(\lambda))$ is not constant, and we identify M with

$\mathbb{D}(\lambda_0, 1) \subset \mathbb{C}$ in the rest of the proof. By assumption $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$ and we may assume without loss of generality that $\varphi_{\lambda_0} = \psi_{\lambda_0} = \mathrm{Id}$ and hence $f = f_{\lambda_0}$. Let $x := f_{\lambda_0}^n(v_{\lambda_0})$, and hence, by assumption, $x \in \partial W$.

Let $N \in \mathbb{N}$ be such that f (and therefore each map f_λ) have the N-island property. Since by assumption $\gamma(\lambda_0)$ has infinitely many preimages, let $z_0, \ldots, z_N$ denote $N+1$ such distinct preimages. We apply Lemma 2.16 to the map $F(\lambda, z) := f_\lambda(z)$ and to the z_i, thus there exists a branched cover $\pi : \mathbb{D} \to V$, with $\pi(0) = \lambda_0$, where V is a neighborhood of λ_0, and holomorphic maps $t \mapsto x_i(t)$ on $\mathbb{D}$ such that $f_{\pi(t)}(x_i(t)) = \gamma \circ \pi(t)$, and $x_i(0) = z_i$. In other words, up to replacing the family $\{f_\lambda\}_{\lambda \in V}$ by the family $\{f_{\pi(t)}\}_{t \in \mathbb{D}}$, we may assume that each preimage $z_i(\lambda)$ moves holomorphically, satisfying $f_\lambda(z_i(\lambda)) = \gamma(\lambda)$. To keep notations light, we will still denote this reparametrized family by $\{f_\lambda\}_{\lambda \in V}$. Let D_i $(0 \leq i \leq N+1)$ be Jordan domains with pairwise disjoint closures each containing z_i, and let $\delta > 0$ be small enough that for all $0 \leq i \leq N$ and $\lambda \in \mathbb{D}(\lambda_0, \delta)$, we have $z_i(\lambda) \in D_i$.

Decreasing δ if necessary, the function $G(\lambda) := \psi_\lambda^{-1}(f_\lambda^n(v_\lambda))$ is open on $\mathbb{D}(\lambda_0, \delta)$ by Lemma 2.15, and $G(\lambda_0) = x \in \partial W$. It follows that $G(\mathbb{D}(\lambda_0, \delta))$ contains a disk $\Delta \subset X$ of d_X-radius say $\epsilon > 0$ centered at x. By the island property, there exists $0 \leq i \leq N$ and $U \Subset \Delta \cap W$ such that $f : U \to D_i$ is a conformal isomorphism. Up to relabeling, we will assume without loss of generality that $i = 0$.

Since U is contained in the image of G, we let V_1 denote a connected component of $G^{-1}(U)$ inside $\mathbb{D}(\lambda_0, \delta)$. If D_0 (and therefore U) is small enough, then $\overline{V_1}$ is a Jordan domain as well. Let us now define $H(\lambda) := f_\lambda^{n+1}(v_\lambda)$. Our goal is to show that $\overline{z_0(V_1)} \subset H(V_1)$ so that Lemma 2.18 applied to the maps $\lambda \mapsto z_0(\lambda)$ and H gives the result.

In order to see this we write

$$f_\lambda^{n+1}(v_\lambda) = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda) = \varphi_\lambda \circ f_{\lambda_0} \circ G(\lambda),$$

and therefore

$$H(V_1) = \varphi_\lambda(f_{\lambda_0}(G(V_1))) = \varphi_\lambda(f_{\lambda_0}(U)) = \varphi_\lambda(D_0).$$

Now since δ can be taken arbitrarily small, the values of λ can be arbitrarily close to λ_0 and therefore φ_λ is arbitrarily close to the identity. It follows that $H(V_1) = \varphi_\lambda(D_0) \simeq D_0$, while $\overline{z_0(V_1)} \subset z_0(\overline{\mathbb{D}(\lambda_0, \delta)}) \subset D_0$. Moreover, $\partial z_0(\overline{\mathbb{D}(\lambda_0, \delta)})$ separates the boundaries of these two sets, so the hypotheses of Lemma 2.18 can be applied. This gives the existence of λ' arbitrarily close to λ_0 such that $f_{\lambda'}^{n+1}(v_{\lambda'}) = z_0(\lambda')$, and since $f_{\lambda'}(z_0(\lambda')) = \gamma(\lambda')$, we do have $f_{\lambda'}^{n+2}(v_{\lambda'}) = \gamma(\lambda')$.

□

With the tools above, we can give some additional characterizations of activity of singular values, which will be usefull to prove approximation theorems in the next section.

Proposition 2.19 (Active singular values). *A singular value $v(\lambda)$ is active at λ_0 if and only if one of the following three cases occurs:*

(1) There exists $n \geq 0$ such that $f_{\lambda_0}^n(v(\lambda_0)) \in \partial W_{\lambda_0}$, non persistently.

(2) There exists an injective sequence of parameters $\lambda_k \to \lambda_0$, such that for some sequence of integers $n_k \to \infty$,

$$f_{\lambda_k}^{n_k}(v(\lambda_k)) \in \partial W_{\lambda_k}.$$

(3) There exists a neighborhood U of λ_0 such that the family $(\lambda \mapsto f_\lambda^n(v_\lambda))_{n \in \mathbb{N}}$ is well defined and not normal in U. This case can only occur if f is exceptional.

Proof of Proposition 2.19. Taking the formal negation of the definition of passivity, we obtain that $v(\lambda)$ is active at λ_0 if and only if both of the following conditions are satisfied for all neighborhood V of λ_0:

(a) for all $n \geq 0$, there exists $\lambda \in V$ such that $f_\lambda^n(v_\lambda) \in W_\lambda$; and
(b) the family of holomorphic maps $\{\lambda \mapsto f_\lambda^n(v_\lambda)\}$ is either not well-defined on V, or it is well-defined but non-normal.

It is clear that condition (3) implies both (a) and (b), and that conditions (1) and (2) each imply (b). Let us prove that (1) also implies (a). Assume that $f_{\lambda_0}^n(v(\lambda_0)) \in \partial W_{\lambda_0}$ non-persistently. Let $G(\lambda) := \psi_\lambda^{-1} \circ f_\lambda^n(v(\lambda))$. Since $W_\lambda := \psi_\lambda(W)$, we have $G(\lambda) \in W \Leftrightarrow f_\lambda^n(v(\lambda)) \in W_\lambda$, and $G(\lambda) \in \partial W \Leftrightarrow f_\lambda^n(v(\lambda)) \in \partial W_\lambda$. By Lemma 2.15, the map G is either open or constant near λ_0; and $G(\lambda)$ is non-constant since by assumption $f_{\lambda_0}^n(v(\lambda_0)) \in \partial W_{\lambda_0}$ non-persistently. Therefore, there exists $\lambda \in V$ such that $G(\lambda) \in W$, hence $f_\lambda^n(v(\lambda)) \in W_\lambda$. Now that we know that (1) implies (a), it is clear that (2) also implies (a). We have therefore proved that $v(\lambda)$ is active at λ_0 if one of the three cases (1), (2) or (3) occurs. Let us now prove that case (3) can only occur if f_{λ_0} is exceptional.

Suppose that (3) holds. Then X cannot be hyperbolic by Lemma 2.7, and therefore $X = \mathbb{P}^1$ or a complex torus. But endomorphisms of complex tori have no singular values by Hurwitz's formula, so this last possibility is in fact excluded; therefore, $X = \mathbb{P}^1$.

Assume for a contradiction that f_{λ_0} is not exceptional. Then in particular $\bigcup_{n \geq 0} f_{\lambda_0}^{-n}(\partial W)$ is infinite, by Lemma 2.8, so there exists $z_1, z_2, z_3 \in \mathbb{P}^1$ three distinct points such that $f_{\lambda_0}^N(x_1) = f_{\lambda_0}^N(x_2) = f_{\lambda_0}^N(x_3) =: y \in \partial W_{\lambda_0}$ for some $N \geq 1$. Let $D \subset M$ be a one-dimensional disk passing through λ_0 such that $\{\lambda \mapsto f_\lambda^n(v_\lambda) : n \in \mathbb{N}\}$ is still well-defined but non-normal on D. By Lemma 2.16 applied with $F(\lambda, z) := f_\lambda^N(z)$ and $\gamma(\lambda) := \psi_\lambda(y)$, there exists a neighborhood V of λ_0 in D, a branched cover $\pi : \mathbb{D} \to V$ and holomorphic maps $x_i : \mathbb{D} \to \mathbb{P}^1$ such that for all $t \in \mathbb{D}$,

$$f_{\pi(t)}^N(x_i(t)) = \psi_{\pi(t)}(y).$$

The family $\{t \mapsto f_{\pi(t)}^n(v_{\pi(t)}) : n \in \mathbb{N}\}$ is non-normal on $\mathbb{D}$, so by Montel's theorem it cannot omit the three moving values $x_1(t), x_2(t), x_3(t)$; therefore, there exists $t_1 \in \mathbb{D}$ and $n \in \mathbb{N}$ such that say $f_{\pi(t_1)}^n(v_{\pi(t_1)}) = x_1(t_1)$, which means that $f_{\lambda_1}^{N+n}(v_{\lambda_1}) \in \partial W_{\lambda_1}$, where $\lambda_1 := \pi(t_1)$. But this contradicts the assumption that $\{\lambda \mapsto f_\lambda^n(v_\lambda) : n \in \mathbb{N}\}$ is well-defined on D, hence on V. Therefore, f is exceptional.

Conversely, assume that both (a) and (b) hold. There are two possibilities: first, if there exists a neighborhood V such that $\{\lambda \mapsto f_\lambda^n(v_\lambda)\}$ is well-defined but not normal, then we are in case (3). Assume from now on that this is not the case, and let $G_n(\lambda) := \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda)$ as above. Then, for all neighborhood V of λ_0, there exists $n \in \mathbb{N}$ such that $G_n(V) \cap W_{\lambda_0} \neq \varnothing$ (by (a)) and $G_n(V) \cap (X \setminus W_{\lambda_0}) \neq \varnothing$ (because $\{\lambda \mapsto f_\lambda^n(v_\lambda)\}$ is not well defined), so that $G_n(V) \cap \partial W_{\lambda_0} \neq \varnothing$. By considering a basis of neighborhoods $(V_k)_{k \in \mathbb{N}}$ of λ_0, we obtain a sequence $\lambda_k \to \lambda_0$ (not necessarily injective) and a sequence of integers $(n_k)_{k \in \mathbb{N}}$ (not necessarily unbounded) such that $f_{\lambda_k}^{n_k}(v_{\lambda_k}) \notin W_{\lambda_k}$. If the sequence (n_k) is bounded, then up to extraction it is constant equal to some $N \in \mathbb{N}$; and by continuity we have $f_{\lambda_0}^N(v_{\lambda_0}) \in \partial W$. By (a) this relation is not persistent on M, and so we are in case (1).

If the sequence (n_k) is unbounded, then up to extraction we can assume that it is strictly increasing. Then the sequence (λ_k) must be injective, and so we are in case (2).

□

2.4. **Density Theorems.** In this section we prove that given the activity locus $\mathcal{A}(v_\lambda)$ of a singular value v_λ, parameters for which v_λ has a Misiurewicz relation and parameters for which the orbit of v_λ lands on the boundary of W_λ are dense in $\mathcal{A}(v_\lambda)$. We also show that $\mathcal{A}(v_\lambda)$ is nowhere locally contained in a proper analytic subset of M.

Definition 2.20 (Misiurewicz relation)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of holomorphic maps, and $\lambda_0 \in M$. We say that f_{λ_0} has a Misiurewicz relation if there exists a singular value v_{λ_0}, $n \in \mathbb{N}$ and a repelling periodic point z_{λ_0} such that $f^n_{\lambda_0}(v_{\lambda_0}) = z_{\lambda_0}$.*

We say that a Misiurewicz relation is persistent if it holds on a parameter neighborhood of λ_0.

Lemma 2.21 (Misiurewicz relations imply activity)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of Ahlfors island maps. Let $\lambda_0 \in M$ be such that f_{λ_0} has a Misiurewicz relation, i.e. there exists $v_{\lambda_0} \in S(f_{\lambda_0})$ and $n \in \mathbb{N}$ such that $f^n_{\lambda_0}(v_{\lambda_0})$ is a repelling periodic point, and this relation is not persistent. Then v_λ is active at λ_0.*

Proof. By definition of activity, we may assume without loss of generality that there exists a neighborhood V of λ_0 on which $\{\lambda \mapsto f^k_\lambda(v_\lambda) : k \in \mathbb{N}\}$ are well-defined (otherwise, v_λ is active at λ_0 and we are done). Then the proof is the same as in the classical case of e.g. rational maps. We reproduce it here for the convenience of the reader.

Let p denote the period of the repelling cycle. There exists a neighborhood U of λ_0 such that the repelling periodic point $z_{\lambda_0} = f^n_{\lambda_0}(v_{\lambda_0})$ moves holomorphically over U as $\lambda \mapsto z_\lambda$, and remains repelling. Moreover, there exists $r > 0$ such that the cycle of z_λ is linearizable on $\mathbb{D}(z_\lambda, r)$, that is, there exists local biholomorphisms $\zeta_\lambda : \mathbb{D}(0,r) \to W_\lambda$ depending holomorphically on λ, such that $\zeta_\lambda(0) = z_\lambda$ and $f^p_\lambda \circ \zeta_\lambda(z) = \zeta_\lambda(\rho_\lambda z)$, where ρ_λ is the multiplier of the repelling cycle. Let $u(\lambda) := \zeta_\lambda^{-1}(f^n_\lambda(v_\lambda))$. Then $f^{n+kp}_\lambda(v_\lambda) = f^{kp}_\lambda \circ \zeta_\lambda(u(\lambda)) = \zeta_\lambda(\rho^k_\lambda u(\lambda))$. But since by assumption, $u(\lambda_0) = 0$ and u does not vanish identically and $|\rho_\lambda| > 1$, it is clear that the family $\{\lambda \mapsto f^{n+kp}_\lambda(v_\lambda)\}_{k\in\mathbb{N}}$ is not normal at λ_0, hence that v_λ is indeed active at λ_0. □

Lemma 2.22 (Activity loci are not contained in analytic subsets)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of Ahlfors island maps, and let $\mathcal{A}(v_\lambda)$ be the activity locus of a singular value v_λ. Then $\mathcal{A}(v_\lambda)$ is nowhere locally contained in a proper analytic subset of M. More precisely, if $\lambda_0 \in \mathcal{A}(v_\lambda) \cap H$, where $H \subset M$ is a proper analytic subset, then for every neighborhood U of λ_0 in M, $U \cap (\mathcal{A}(v_\lambda) \setminus H) \neq \varnothing$.*

Proof. Let $\lambda_0 \in \mathcal{A}(v_\lambda)$, H a proper analytic subset of M containing λ_0, and U be a small polydisk centered at λ_0 in M. Assume for a contradiction that $\mathcal{A}(v_\lambda) \cap U \subset H \cap U$. Let $h_n(\lambda) := f^n_\lambda(v_\lambda)$, wherever this expression is well-defined. Let z_{λ_0} be a repelling periodic point of period at least 3 for f_{λ_0} which is not Picard exceptional. Let z_λ be the corresponding repelling periodic point for f_λ given by the Implicit Function Theorem. Up to reducing U, we may assume that $\lambda \mapsto z_\lambda$ is defined over U.

Since $\lambda_0 \in \mathcal{A}(v_\lambda)$, there is no $N \in \mathbb{N}$ such that $h_N(\lambda) \in X \setminus W_\lambda$ for all $\lambda \in U$. By Lemma 2.14 and the assumption that $v(\lambda)$ is active at λ_0, we cannot have that $f^n_\lambda(v_\lambda) \in \partial W_\lambda$ for all λ in an open subset of U. Therefore, if $\lambda \in U$ and $n \in \mathbb{N}$ are such that $h_n(\lambda) \in \partial W_\lambda$, then $\lambda \in \mathcal{A}(v_\lambda) \cap U$, therefore in H.

We now distinguish two cases:

(1) either there exists $n_0 \in \mathbb{N}$ and $\lambda_1 \in U$ such that $h_{n_0}(\lambda_1) \in \partial W_{\lambda_1}$ non-persistently;
(2) or for every $n \in \mathbb{N}$, h_n is well-defined over U but not normal.

Let us first treat case (1). Let D be a one-dimensional holomorphic disk passing through λ_1 and not contained in H. Then by the choice of λ_1 and our previous observation, $h_{n_0}(\lambda_1) \in \partial W_{\lambda_1}$ and there exists $\lambda \in D \setminus \{\lambda_1\}$ such that $h_n(\lambda) \notin \partial W_\lambda$. By the Shooting Lemma (Proposition 2.17) applied with $\gamma(\lambda) := z_\lambda$ and $M := D$, we find some $\lambda_2 \in D \setminus \{\lambda_1\}$ such that $h_{n_0+1}(\lambda_2) = z_{\lambda_2}$, in other words, v_{λ_2} is Misiurewicz. Therefore $\lambda_2 \in \mathcal{A}(v_{\lambda_2})$, but $\lambda_2 \notin H$, a contradiction.

Case 2 follows from a similar but more classical application of Montel's theorem. □

Proposition 2.23 (Density of truncated parameters). *Assume that f_{λ_0} is a non-exceptional Ahlfors island map. Let v_λ be a singular value, and assume that it is active at λ_0. Then there exists $n_k \to +\infty$ and $\lambda_k \to \lambda_0$ such that $f^{n_k}(v_{\lambda_k}) \in \partial W_{\lambda_k}$ non persistently.*

Proof. Given $N \in \mathbb{N}$, we will construct λ arbitrarily close to λ_0 such that $f_\lambda^n(v_\lambda) \in \partial W_\lambda$, for some $n \geq N$. In view of Lemma 2.10 and since f is not exceptional, the set $\bigcup_{n\geq 0} f^{-n}(\partial W)$ is infinite. By the Ahlfors Island property we may find $n \geq N$ and $x \in f^{-n}(\partial W)$ with infinitely many preimages. By Lemma 2.16 applied to $F(\lambda, z) := f_\lambda^n(z)$ and $\gamma(\lambda) := \psi_\lambda \circ f^n(x) \in \partial W_\lambda$, up to passing to a branched cover in parameter space, we may assume without loss of generality that there is a local holomorphic germ $\lambda \mapsto x_\lambda$, with $x_{\lambda_0} = x$ and for all λ close enough to λ_0, $f_\lambda^n(x_\lambda) \in \partial W_\lambda$.

By Proposition 2.19 and since v_λ is active at λ_0, we may assume without loss of generality that there is $n_0 \in \mathbb{N}$ such that $f_{\lambda_0}^{n_0}(v_{\lambda_0}) \in \partial W_{\lambda_1}$ non-persistently. Applying Lemma 2.17 with $\gamma(\lambda) := x_\lambda$, we find λ_1 arbitrarily close to λ_0 such that $f_{\lambda_1}^{n_0+2}(v_{\lambda_1}) = x_{\lambda_1}$, which implies $f_{\lambda_1}^{n_0+2+n}(v_{\lambda_1}) = y_{\lambda_1} \in \partial W_{\lambda_1}$.

□

Proposition 2.24 (Density of Misiurewicz parameters). *Let v_λ be a singular value, and assume that it is active at λ_0. Let x_{λ_0} be a repelling periodic point for an Ahlfors island map f_{λ_0} of period at least 3 which is not a Picard exceptional value, and let x_λ be its analytic continuation in some neighborhood of λ_0. Then there is $n_k \to +\infty$ and $\lambda_k \to \lambda_0$ such that*

$$f_{\lambda_k}^{n_k}(v_{\lambda_k}) = x_{\lambda_k}.$$

Proof. Let us first assume that f is not exceptional. Let $\epsilon > 0$ and $N \in \mathbb{N}$, and let $\lambda \mapsto x_\lambda$ denote the holomorphic motion of x_{λ_0} as a repelling periodic point, which we may assume to be well-defined for $\lambda \in \mathbb{B}(\lambda_0, \epsilon)$ (up to taking $\epsilon > 0$ small enough). By Proposition 2.23, there exists $\lambda_1 \in \mathbb{B}(\lambda_0, \frac{\epsilon}{2})$ and $n_1 > N$ such that $f_{\lambda_1}^N(v_{\lambda_1}) \in \partial W_{\lambda_1}$ (non-persistently). By Proposition 2.17, there exists $\lambda_2 \in \mathbb{B}(\lambda_1, \frac{\epsilon}{2})$ such that $f_{\lambda_2}^{n_1+2}(v_{\lambda_2}) = x_{\lambda_2}$, and we are done.

Asssume now that f is exceptional. By Proposition 2.6, either for all $\lambda \in M$ we have that f_λ is exceptional, or the set of $\lambda \in M$ such that f_λ is exceptional is a proper analytic subset of M. In the latter case, we may use Lemma 2.22 to perturb slightly λ_0 to remain in the activity locus of v_{λ_0} but outside this analytic set, thus reducing to the non-exceptional case. Finally, if all maps f_λ are exceptional, then we can just apply the classical argument using Montel's theorem. □

3. Characterization of stability: Proof of Theorem 1.5

In this section we prove that the backward orbits of repelling periodic points move holomorphically, provided they do not collide with the postsingular set. We then use this fact to prove Theorem 1.5.

In what follows M always denotes a connected complex manifold.

Definition 3.1 (Holomorphic motions respecting the dynamics). A *holomorphic motion* of a set $A \subset X$ over an open set $U \subset M$ with basepoint $\lambda_0 \in U$ is a map $H : U \times A \to X$ given by $(\lambda, x) \mapsto H_\lambda(x)$ such that

(1) for each $x \in A$, $\lambda \mapsto H_\lambda(x)$ is holomorphic ,
(2) for each $\lambda \in U$, $H_\lambda(\cdot)$ is injective on A, and,
(3) $H_{\lambda_0} \equiv \mathrm{Id}$.

A holomorphic motion of a set X *respects the dynamics* of the holomorphic family $\{f_\lambda\}_{\lambda \in M}$ if

$$H_\lambda \circ f_{\lambda_0} = f_\lambda \circ H_\lambda$$

whenever both x and $f_{\lambda_0}(x)$ belong to A.

Definition 3.2 (J-stability). Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of Ahlfors maps. Given $\lambda_0 \in M$ the map f_{λ_0} is J -stable if there exists a neighbourhood U of λ_0 over which the Julia sets move holomorphically, and the holomorphic motion respects the dynamics.

Let

$$\mathcal{O}^-(w, g) := \bigcup_{n \geq 0} g^{-n}(\{w\})$$

denote the backwards orbit of w under the map g.

We define the postsingular set of f_λ as $\bigcup_{n \geq 0} f_\lambda^n(S(f_\lambda))$ (without taking the closure).

Proposition 3.3 (Holomorphic motion of backward orbits). *Let $\{f_\lambda : W_\lambda \to X\}_{\lambda \in M}$ be a natural family of Ahlfors island maps. Let z_0 be a repelling periodic point of period $p \geq 1$ for $f := f_{\lambda_0}$. Let U be a simply connected neighborhood of λ_0 over which the analytic continuation of z_0, denoted by $z_0(\lambda)$, remains repelling, and suppose that for all $\lambda \in U$, f_λ has no non-persistent Misiurewicz relations of the form $f_\lambda^n(v(\lambda)) = z_0(\lambda)$, where $v(\lambda)$ is either a critical or asymptotic value. Then, there is a holomorphic motion*

$$\begin{array}{rccc} H: & U \times \mathcal{O}^-(z_0, f^p) & \longrightarrow & \mathcal{O}^-(z_0(\lambda), f_\lambda^p) \\ & (\lambda, z) & \longmapsto & z(\lambda) \end{array}$$

preserving the dynamics of f_λ^p.

Proof. Let $\varphi_\lambda, \psi_\lambda : X \to X$ be the quasiconformal homeomorphisms such that $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$ (see Definition 1.3). We will prove the statement in several steps. We shall first show that for every choice of $n \geq 1$, the set $f_\lambda^{-n}(\{z_0\})$ moves holomorphically with $\lambda \in U$. Then, we will prove that there are no collisions between the motion of points belonging to $f_\lambda^{-m}(\{z_0\})$ and $f_\lambda^{-k}(\{z_0\})$ when $k \neq m$ provided both are multiples of p. For $n = 1$ consider the set

$$Z_1 = \{(\lambda, z) \mid f_\lambda(z) = z_0(\lambda)\},$$

which is an analytic hypersurface of $U \times X$. Let

$$\pi_1 : Z_1 \to M$$

denote the projection onto the first coordinate.

Step 1. We first claim that every irreducible component of Z_1 is the graph of a holomorphic map from U to X.

We will first treat the case where $z_0 \in S(f)$. Let $(\lambda_1, z_1) \in Z_1$. By assumption, Misiurewicz relations (if there are any) are persistent, and $f_{\lambda_0}^p(z_0) = z_0$ is such a relation. Therefore for all $\lambda \in U$, $z_0(\lambda) \in S(f_\lambda)$ and, since f_λ is a natural family, $z_0(\lambda) = \varphi_\lambda(z_0)$. Let $z_1(\lambda) := \psi_\lambda \circ \psi_{\lambda_1}^{-1}(z_1)$. Then, for all $\lambda \in U$:

$$\begin{aligned} f_\lambda(z_1(\lambda)) &= \varphi_\lambda \circ f \circ \psi_\lambda^{-1} \circ \psi_\lambda \circ \psi_{\lambda_1}^{-1}(z_1) \\ &= \varphi_\lambda \circ \varphi_{\lambda_1}^{-1} \circ \varphi_{\lambda_1} \circ f \circ \psi_{\lambda_1}^{-1}(z_1) \\ &= \varphi_\lambda \circ \varphi_{\lambda_1}^{-1} \circ f_{\lambda_1}(z_1) \\ &= \varphi_\lambda \circ \varphi_{\lambda_1}^{-1}(z_0(\lambda_1)) \\ &= \varphi_\lambda(z_0) = z_0(\lambda). \end{aligned}$$

Therefore, for all $\lambda \in U$, $(\lambda, z_1(\lambda)) \in Z_1$, which proves Step 1 in the case where $z_0 \in S(f)$.

We now assume (in the rest of the proof of Step 1) that $z_0 \notin S(f)$. Let $(\lambda_1, z_1) \in Z_1$ and let Z denote the irreducible component of Z_1 containing (λ_1, z_1). Again by the persistence assumption, $z_0(\lambda_1)$ is not a critical value, and hence z_1 is not a critical point of f_{λ_1}. By the Implicit Function Theorem, Z is then a complex manifold and $\pi_1 : Z \to U$ is locally invertible. It is then well known that π_1 is a covering unless it has as asymptotic value.

So suppose that λ^* is an asymptotic value of π_1, i.e. there exists a path $(\lambda_t, z_t) \in Z_1$, $t \in [0,1)$ such that $\lambda_t \to \lambda^*$ while $z_t \to \partial W_{\lambda^*}$, as $t \to 1$. Let φ_t, ψ_t, f_t denote respectively $\varphi_{\lambda(t)}, \psi_{\lambda(t)}, f_{\lambda(t)}$. Then, by definition of Z_1,

$$f_t(z_t) = (\varphi_t \circ f \circ \psi_t^{-1})(z_t) = z_0(\lambda(t)),$$

and hence

$$f(\psi_t^{-1}(z_t)) = \varphi_t^{-1}(z_0(\lambda(t))). \tag{2}$$

Now, when $t \to 1$ we have that $z_t \to \partial W_{\lambda^*}$, hence $\psi_t^{-1}(z_t) \to \partial W_{\lambda_0}$, since $\psi_{\lambda^*}(W_0) = W_{\lambda^*}$. On the other hand, since $z_0(\lambda(t)) \to z_0(\lambda^*)$ it follows that $\varphi_t^{-1}(z_0(\lambda(t))) \to \varphi_{\lambda^*}^{-1}(z_0(\lambda^*))$, which makes this point an asymptotic value of f by (2), because $t \mapsto \psi_t^{-1}(z_t))$ is a curve converging to ∂W_0. Hence $z_0(\lambda^*)$ is an asymptotic value for f_{λ^*}, contradiction. Thus $\pi_1 : Z \to U$ is a covering map, and since U is simply connected, it is invertible, which implies that Z is a holomorphic graph above U. This concludes the proof of Step 1.

Step 2. The irreducible components of Z_1 are pairwise disjoint.

Indeed, let us assume for a contradiction that Z and Z' are two distinct irreducible components of Z_1 and $(\lambda_1, z_1) \in Z \cap Z'$. Then z_1 must be a critical point for f_{λ_1}, and $z_0(\lambda_1)$ must be a critical value. But then by the proof of Step 1, both Z and Z' are the graph of the same map $\lambda \mapsto \psi_\lambda \circ \psi_{\lambda_1}^{-1}(z_1)$, so $Z = Z'$, a contradiction.

Step 3: Conclusion. The set $\mathcal{O}^-(z_0(\lambda), f_\lambda^p) = \bigcup_{n \geq 0} f_\lambda^{-np}(\{z_0(\lambda)\})$ moves holomophically over U.

Steps 1 and 2 prove that the set Z_1 is a disjoint union of holomorphic graphs $\Gamma_i = \{(\lambda, z_i(\lambda)), \lambda \in U\}$ over U, i.e. that the set $f_\lambda^{-1}(\{z_0(\lambda)\})$ moves holomorphically over U, with $H_\lambda(z_i(\lambda_0)) := z_i(\lambda)$.

Now assume we have proven the existence of a holomorphic motion of $f_\lambda^{-n+1}(\{z_0(\lambda)\})$. By considering $z_n \in f^{-n}(\{z_0\})$ and applying the same arguments, we obtain by induction that for every $n \in \mathbb{N}$,

$$Z_n := \{(\lambda, z) \mid f_\lambda^n(z) = z_0(\lambda), \text{but } f_\lambda^j(z) \neq z_0(\lambda) \text{ for any } j < n\}$$

is a disjoint union of graphs over U, hence also moves holomorphically over U. Observe that by construction, this holomorphic motion preserves the dynamics.

To end the proof we need to show that if $n_1 \neq n_2$ then $Z_{n_1 p}$ and $Z_{n_2 p}$ are disjoint sets. Let $g_\lambda := f_\lambda^p$ so that $z_0(\lambda)$ is a fixed point of g_λ.

So suppose $z(\lambda)$ and $\tilde{z}(\lambda)$ satisfy the defining equation

$$g_\lambda^{n_1}(z(\lambda)) = g_\lambda^{n_2}(\tilde{z}(\lambda)) = z_0(\lambda),$$

for some $n_1, n_2 \in \mathbb{N}$ and assume they coincide at some $\lambda = \lambda^*$, i.e. $z(\lambda^*) = \tilde{z}(\lambda^*) = z^*$.

Let $n = \max(n_1, n_2)$. Since $z_0(\lambda)$ is fixed for g_λ, we have that

$$g_\lambda^n(z(\lambda)) = g_\lambda^n(\tilde{z}(\lambda)) = z_0(\lambda),$$

for all $\lambda \in U$. Then, either $z(\lambda) \equiv \tilde{z}(\lambda)$ in U and we are done, or z^* is a critical point of $g_{\lambda^*}^n$, in which case we would have a non-persistent Misiurewicz relation, impossible by assumption. □

Corollary 3.4 ($J-$stability). *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of Ahlfors island map. Let $U \subset M$ be a simply connected domain over which a repelling periodic point $z(\lambda)$ moves holomorphically, and assume that there are no non-persistent Misiurewicz relations on U. Then the family $\{f_\lambda\}_{\lambda \in M}$ is J-stable on a neighborhood of λ_0.*

Proof. By Lemma 2.10, the set $\mathcal{O}^-(z_0, f^p)$ is dense in $J(f)$. Therefore, by the classical λ-lemma, this holomorphic motion extends to a holomorphic motion of $J(f)$

$$\begin{array}{rccc} H: & U \times J(f) & \longrightarrow & J(f_\lambda) \\ & (\lambda, z) & \longmapsto & H_\lambda(z) := z(\lambda) \end{array}$$

which by continuity still preserves the dynamics of f^p, i.e.

$$H_\lambda \circ f^p(z) = f_\lambda^p \circ H_\lambda(z)$$

for all $z \in J(f)$. We claim that we must then have in fact

$$H_\lambda \circ f(z) = f_\lambda \circ H_\lambda(z)$$

for all $z \in J(f)$. By Theorem 2.2 and continuity, it is enough to prove this only for repelling periodic points. Let x be a repelling periodic point of period m for $f = f_{\lambda_0}$. Reducing U if necessary, let x_λ, $\lambda \in U$, denote its local analytic continuation as a repelling periodic point of period m for f_λ (given by the Implicit Function Theorem).

Since $f^{mp}(x) = x$, we have $f_\lambda^{mp} \circ H_\lambda(x) = H_\lambda(x)$ for all $\lambda \in U$ and since H is a holomorphic motion with basepoint λ_0, we have $H_{\lambda_0}(x) = x$. By continuity of H and since repelling fixed points of f_λ^{mp} are isolated and move holomorphically, we must have $H_\lambda(x) = x_\lambda$ locally for λ close to λ_0, and therefore

$$H_\lambda \circ f(z) = f_\lambda \circ H_\lambda(z)$$

for all repelling periodic points of f (hence for all $z \in J(f)$ by continuity) and for all λ in a neighborhood of λ_0. The result finally follows from the Identity Theorem applied on M to the holomorphic maps $\lambda \mapsto H_\lambda \circ f(z)$ and $\lambda \mapsto f_\lambda \circ H_\lambda(z)$, $z \in J(f)$. □

Proposition 3.5. *Let $\{f_\lambda\}_{\lambda \in M}$ denote a natural family of Ahlfors island maps. Assume that this family is J-stable; then all singular values are passive on U.*

Proof. Let $\lambda_0 \in M$, and let $h_\lambda : J(f_{\lambda_0}) \to J(f_\lambda)$ denote the dynamical holomorphic motion of the Julia set.

Let $v \in S(f_{\lambda_0})$ and $v_\lambda := \varphi_\lambda(v)$. Let z be a repelling periodic point of f_{λ_0} of period at least 3, with infinitely many preimages and which is not in the forward orbit of v. (Such points always exist by the island property). Assume for a contradiction that v_λ is active at λ_0. By Proposition 2.24, there exists $\lambda_1 \in M$ close to λ_0 and $n \in \mathbb{N}$ such that $f^n_{\lambda_1}(v_{\lambda_1}) = h_{\lambda_1}(z)$.

But since $h_{\lambda_1} : J(f_{\lambda_0}) \to J(f_{\lambda_1})$ is a topological conjugacy, this is not possible. Therefore, v_λ must be passive on M. □

Proof of Theorem 1.5 . Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of Ahlfors maps. Proposition 3.5 proves that J-stability implies passivity of all singular values.

Let us prove that conversely, if $\lambda_0 \in M$ and if there is a neighborhood V of λ_0 such that all critical and asymptotic values are passive on V, then there is a neighborhood $U \subset V$ of λ_0 such that $\{f_\lambda\}_{\lambda\in U}$ is J-stable.

Let $z_0(\lambda_0)$ denote a repelling periodic point of period $p \geq 1$ for f_{λ_0}. Let $U \subset V$ denote a simply connected neighborhood λ_0 on which $z_0(\lambda)$ moves holomorphically as a repelling periodic point of f_λ.

By Lemma 2.21, none of the maps f_λ, $\lambda \in U$, may have any non-persistent Misiurewicz relation. We can therefore apply Corollary 3.4, which asserts that $\{f_\lambda\}_{\lambda\in U}$ is indeed J-stable. □

Remark 3.6. The proof given implies the following: if all critical and asymptotic values are passive, then all singular values are passive. More generally, using Proposition 2.19 one could prove directly that the set of active singular values at a given parameter $\lambda_0 \in M$ is closed.

4. Finite type maps and attracting cycles

We consider, as above, a natural family $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1} : W_\lambda \to X$ of finite type maps. In this section we prove some results that are necessary for Theorem 1.6, but which are also of independent interest.

In the context of [ABF21], where we dealt with natural families of meromorphic maps, i.e. $X = \mathbb{P}^1$ and $\partial W_\lambda = \{\infty\}$ for all $\lambda \in M$, we were able to prove (see Theorem B, the Accessibility Theorem in [ABF21]) that certain active parameter values (those involving asymptotic values mapping eventually to infinity), say λ_0, can always be accessed by a curve of parameters $\lambda(t)$ such that $f_{\lambda(t)}$ possesses an attracting cycle, whose multiplier converges to 0 as $\lambda(t)$ tends to λ_0. Despite λ_0 being in the bifurcation locus, this property granted them the name of *virtual centers*, in a analogy to the centers of hyperbolic components in rational maps.

The results we prove in this section are the best possible generalization of this property. More precisely, we shall find a *sequence* of parameters (instead of a curve) with attracting cycles of the same period and arbitrarily small multiplier (Theorem 4.4). The difficulty arising in the new contest of finite type maps is that one must account for the possibility of a tract which accumulates in a complicated way on ∂W, something which cannot happen for meromorphic maps. For these reason, we shall need a more elaborate definition of what was then called a *virtual cycle*.

4.1. **Creation of attracting cycles near virtual cycles.**

Definition 4.1 (Simple virtual cycle). *Let $f : W \to X$ be a finite type map, and let T be a tract above an asymptotic value v. We say that $x \in \partial T \cap \partial W$ is a good point in ∂T if it is in the accumulation locus of an oriented hyperbolic geodesic $\gamma \subset T$ which has no limit point in W.*

If there is $n \in \mathbb{N}$ such that $f^n(v)$ is a good point *in ∂T, we say that $v, f(v), \ldots, f^n(v)$ is a* simple virtual cycle *of length $n+1$.*

Here is an equivalent formulation of the definition above. Let $g : \mathbb{H} \to T$ be a conformal isomorphism between the left half-plane $\mathbb{H}$ and the simply connected tract T, normalized so that $f \circ g(z) \to v$ as $\operatorname{Re} z \to -\infty$. Then $x \in \partial T \cap \partial W$ is a good point if and only if it there is a constant $y_0 \in \mathbb{R}$ and a sequence $t_k \to +\infty$ such that $g(-t_k + iy_0) \to x$. Moreover, up to choosing an appropriate normalization of g, we can assume without loss of generality that $y_0 = 0$.

Remark 4.2. In the case of finite type *meromorphic* maps, $\partial W = \{\infty\}$ so that the only good point is always ∞. Therefore, this definition agrees with the one from [ABF21], in the sense that every simple virtual cycle is a virtual cycle as defined in [ABF21, Definition 1.3] (see also [FK21]).

Definition 4.3 (Non-persistency). *Let $v_{\lambda_0}, f_{\lambda_0}(v_{\lambda_0}), \ldots, f^n_{\lambda_0}(v_{\lambda_0})$ be a simple virtual cycle for some $\lambda_0 \in M$ and $n \in \mathbb{N}$. We say that the cycle is* non persistent, *if $f^n_{\lambda_0}(v)$ maps to ∂W_{λ_0} non-persistently (see Definition 2.13).*

The main theorem in this section is the following.

Theorem 4.4 (Attracting cycles). *Assume that there is $\lambda_0 \in M$ such that f_{λ_0} has a non-persistent simple virtual cycle of length $n+1$. Then there is a sequence $\lambda_k \to \lambda_0$ such that f_{λ_k} has an attracting cycle of period $n+1$, of multiplier $\rho_k \to 0$, which captures the asymptotic value v_{λ_k}.*

For the proof, we will use the following technical lemmas, proved in [ABF21].

Lemma 4.5 (Hyperbolic distance in tracts [ABF21, Lemma 4.1]). *Let $T \subset X$ be a simply connected hyperbolic domain, ρ_T be the hyperbolic density in T with respect to a continuous hermitian metric on X, and let $z, w \in T$. Then*

$$\operatorname{dist}_T(z,w) \geq \frac{1}{2}\left|\ln \frac{\operatorname{dist}(w,\partial T)}{\operatorname{dist}(z,\partial T)}\right|. \tag{3}$$

Lemma 4.6 (Asymptotic derivative of the Riemann map [ABF21, Lemma 4.2]). *Let $\mathbb{H}$ be the left half plane, T be a simply connected hyperbolic domain, $g : \mathbb{H} \to T$ be a Riemann map. Then for every $\alpha > 0$,*

$$\lim_{t \to +\infty} |g'(-t)| e^{\alpha t} = \infty. \tag{4}$$

Lemma 4.7 (Distortion of small disks [ABF21, Lemma 2.9]). *Let $\{\varphi_\lambda\}_{\lambda \in \mathbb{D}}$ be a holomorphic motion of X, with $\varphi_0 = \mathrm{id}$. Let $t \mapsto \lambda(t)$ be a continuous path in $\mathbb{D}$ with $\lim_{t \to +\infty} \lambda(t) = 0$, and $t \mapsto r_t$ a continuous function with $r_t > 0$ and $\lim_{t \to +\infty} r_t = 0$. Let $t \mapsto z_t$ be a path in X and $D_t := \mathbb{D}(z_t, r_t)$. Let $\epsilon > 0$; then for all t large enough:*

$$\mathbb{D}(\varphi_{\lambda(t)}(z_t), r_t^{1+\epsilon}) \subset \varphi_{\lambda(t)}(\mathbb{D}(z_t, r_t)) \subset \mathbb{D}(\varphi_{\lambda(t)}(z_t), r_t^{1-\epsilon})$$

Lemma 4.8. *Let $h_\lambda : \mathbb{D} \to \mathbb{C}$ be a holomorphic family of holomorphic maps, with $\lambda \in M$ a domain of $\mathbb{C}^m$ containing 0 and assume that $h_\lambda(0) \equiv 0$ on M. Let $\lambda_k \to 0$ in M and let $\epsilon_k \to 0$,*

$\epsilon_k > 0$. Let U_k denote the connected component of $h_{\lambda_k}^{-1}(\mathbb{D}(0,\epsilon_k))$ containing 0. There exists $c > 0$ such that for all $k \in \mathbb{N}$ large enough,

$$\mathbb{D}(0, c\epsilon_k) \subset U_k.$$

Proof. Since $h_\lambda(0) \equiv 0$, there exists a constant $C > 0$ such that for all (λ, z) in a neighborhood of $(0,0) \in M \times \mathbb{D}$, we have

$$|h_\lambda(z)| \leq C|z|.$$

This means that for all k large enough, (λ_k, ϵ_k) belongs to that neighborhood and

$$h_{\lambda_k}(\mathbb{D}(0, c\epsilon_k)) \subset \mathbb{D}(0, \epsilon_k)$$

where $c := \frac{1}{C}$. In particular, $\mathbb{D}(0, c\epsilon_k) \subset U_k$. □

The proof of Theorem 4.4 will follow from the next lemma, which we will also need later on.

Lemma 4.9 (Finding attracting cycles)**.** *Let $\lambda_0 \in M$, and let v_{λ_0} be an asymptotic value. Let T be a tract above v_{λ_0}, and let $\Phi : T \to \mathbb{H}$ be a Riemann uniformization of T onto the left half-plane. Assume that there exists $\lambda_k \to \lambda_0$ and $t_k \to +\infty$ such that $f_{\lambda_k}^n(v_{\lambda_k}) = \psi_{\lambda_k} \circ \Phi^{-1}(-t_k)$. Then for all k large enough, f_{λ_k} has an attracting cycle of period $n+1$, of multiplier $\rho_k \to 0$, which captures the asymptotic value v_{λ_k}.*

Proof of Lemma 4.9. To simplify the notations, set $f := f_{\lambda_0}$ and $v := v_{\lambda_0}$. Recall that $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$. Let $V := \mathbb{D}^*(v, r)$ be a punctured disk centered at v disjoint from $S(f)$, , so that $f : T \to V$ is a universal cover.

In particular, $f(z) = v + re^{\Phi(z)}$ for all $z \in T$.

Let $V_\lambda := \varphi_\lambda(V)$ and $T_\lambda := \psi_\lambda(T)$, so that $f_\lambda : T_\lambda \to V_\lambda$ is also an infinite degree universal cover, and let $\Phi_\lambda := \Phi \circ \psi_\lambda^{-1} : T_\lambda \to \mathbb{H}$. Then $\varphi_\lambda^{-1} \circ f_\lambda : T_\lambda \to V$ is a universal cover, and so for all $z \in T_\lambda$,

$$f_\lambda(z) = \varphi_\lambda\left(v + re^{\Phi_\lambda(z)}\right) \tag{5}$$

By assumption, $f_{\lambda_k}^n(v_{\lambda_k}) = \psi_{\lambda_k} \circ \Phi^{-1}(-t_k)$.

Now let $D_{t_k} := \Phi_{\lambda_k}^{-1}(\mathbb{D}(-t_k, \pi)) \subset T_{\lambda_k}$ and let U_{t_k} denote the connected component of $f_{\lambda_k}^{-n}(D_{t_k})$ containing v_{λ_k}. We will prove that for all k large enough, $f_{\lambda_k}(D_{t_k}) \Subset U_{t_k}$, or equivalently, $f_{\lambda_k}^{n+1}(U_{t_k}) \Subset U_{t_k}$; this implies the existence of an attracting fixed point for $f_{\lambda_k}^{n+1}$.

First, let us show that for r small and k large, $f_{\lambda_k}(D_{t_k})$ is contained in a small disk centered at v_{λ_k}, or more precisely,

$$f_{\lambda_k}(D_{t_k}) \subset \mathbb{D}\left(v_{\lambda_k}, e^{-t_k(1-\epsilon)}\right). \tag{6}$$

By (5) we have that for all $z \in \mathbb{H}$,

$$f_\lambda \circ \Phi_\lambda^{-1}(z) = \varphi_\lambda(v + re^z).$$

Since $\mathbb{D}(-t_k, \pi) \subset \{z \in \mathbb{C} : \Re z < -t_k + \pi\}$ we have that

$$f_{\lambda_k}(D_{t_k}) \subset \varphi_{\lambda_k}(\mathbb{D}(v, re^{-t_k+\pi}))$$

Let $\epsilon > 0$. By Lemma 4.7, we have for all k large enough:

$$f_{\lambda_k}(D_{t_k}) \subset \mathbb{D}\left(v_{\lambda_k}, (re^\pi)^{1-\epsilon} e^{-t_k(1-\epsilon)}\right), \tag{7}$$

which for r small implies (6).

Now we show that U_{t_k} contains a disk centered at v_{λ_k} whose radius, for t_k large, is much larger than $e^{-t_k(1-\epsilon)}$.

Let us first estimate $\mathrm{dist}(f^n_{\lambda_k}(v_{\lambda_k}), \partial D_{t_k})$. To lighten the notations, let $g := \Phi^{-1}$; then g is univalent on $\mathbb{H}$ and $D_{t_k} = \psi_{\lambda_k} \circ g(\mathbb{D}(-t_k, \pi))$. By Koebe's theorem, $g(\mathbb{D}(-t_k, \pi))$ contains a disk

$$\mathbb{D}(g(-t_k), C|g'(-t_k)|).$$

Then, by Lemma 4.7,

$$\begin{aligned} D_{t_k} &= \psi_{\lambda_k} \circ g(\mathbb{D}(-t_k, \pi)) \supset \mathbb{D}(\psi_{\lambda_k} \circ g(-t_k), C^{1+\epsilon}|g'(-t_k)|^{1+\epsilon}) \\ &= \mathbb{D}(f^n_{\lambda_k}(v_{\lambda_k}), C^{1+\epsilon}|g'(-t_k)|^{1+\epsilon}). \end{aligned} \tag{8}$$

Recall that U_{t_k} denotes the connected component of $f^{-n}_{\lambda_k}(D_{t_k})$ containing v_{λ_k}, and let $\epsilon_k := C^{1+\epsilon}|g'(-t_k)|^{1+\epsilon}$. Applying Lemma 4.8 to $h_\lambda(z) := f^n_\lambda(z + v_\lambda) - f^n_\lambda(v_\lambda)$ and using (8), we obtain the existence of $c > 0$ such that

$$\mathbb{D}(v_{\lambda_k}, c\epsilon_k) \subset U_{t_k}. \tag{9}$$

Finally, from equations (6) and (9), it is enough to prove that

$$\frac{e^{-t_k(1-\epsilon)}}{|g'(-t_k)|^{1+\epsilon}} \to 0 \quad \text{as } t \to +\infty, \tag{10}$$

which follows from Lemma 4.6. This proves that $f^{n+1}_{\lambda_k}(U_{t_k}) \Subset U_{t_k}$, and the result then follows from Schwartz's lemma. Note that (10) also implies that the multiplier goes to zero as $k \to +\infty$, since the modulus of $U_{t_k} \setminus \overline{f^{n+1}_{\lambda_k}(U_{t_k})}$ tends to infinity. □

We now finish the proof of Theorem 4.4.

Proof of Theorem 4.4. By assumption, there exists $t_k \to +\infty$ such that $\Phi^{-1}(-t_k) \to x \in \partial W_{\lambda_0}$, while $f^n(v) = x$. Now, we wish to find a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in parameter space such that

$$\Phi_{\lambda_k} \circ f^n_{\lambda_k}(v_{\lambda_k}) = -t_k. \tag{11}$$

Let $G(\lambda) := \psi^{-1}_\lambda \circ f^n_\lambda(v_\lambda)$ and recall that $G(\lambda_0) = f^n(v) = x$. Given the definition of Φ_λ, (11) is equivalent to

$$\Phi \circ \psi^{-1}_{\lambda_k} \circ f^n_{\lambda_k}(v_{\lambda_k}) = -t_k, \tag{12}$$

or

$$G(\lambda_k) = \Phi^{-1}(-t_k). \tag{13}$$

Since G is a branched cover over a neighborhood of x by Lemma 2.15, there is such a sequence (not necessarily unique if the local degree of G at λ_0 is more than 1). We finally apply Lemma 4.9 to conclude. □

4.2. **Creation of attracting cycles at active parameters.** We conclude this section with the construction of attracting cycles of high period near parameters with active singular values. We begin with the easier case of critical values.

Proposition 4.10 (Super-attracting cycles are dense in the activity locus of critical values). *Let f_{λ_0} be a non exceptional finite type map. Let v_λ be a critical value, and assume that it is active at λ_0. Assume as well that there exists a critical point $c(\lambda_0) \notin S(f_{\lambda_0})$ such that $f_{\lambda_0}(c_{\lambda_0}) = v_{\lambda_0}$. Then there exists $n_k \to +\infty$ and $\lambda_k \to \lambda_0$ such that v_{λ_k} is in a super-attracting cycle of period n_k.*

Proof. By Proposition 2.23, there exists $n_k \to +\infty$ and $\lambda_k \to \lambda_0$ such that $f_{\lambda_k}^{n_k}(v_{\lambda_k}) \in \partial W_{\lambda_k}$. We then apply Proposition 2.17 to find λ_k' arbitrarily close to λ_k such that $f_{\lambda_k'}^{n_k+1}(v_{\lambda_k'}) = c_{\lambda_k'}$; in other words, $v_{\lambda_k'}$ is a super-attracting periodic point of period $n_k + 1$.

□

Proposition 4.11 (Attracting cycles are dense in the activity locus of asymptotic values). *Let f_{λ_0} be a non exceptional finite type map. Let v_λ be an asymptotic value, and assume that it is active at λ_0. Then there exists $n_k \to +\infty$ and $\lambda_k \to \lambda_0$ such that f_{λ_k} has an attracting cycle of period n_k.*

Proof. Let $\epsilon > 0$ and $N \in \mathbb{N}$: we will find $\lambda_* \in \mathbb{B}(\lambda_0, \epsilon)$ with an attracting cycle of period at least N. By Proposition 2.23, there exists $n_k \to +\infty$ and $\lambda_k \to \lambda_0$ such that $f_{\lambda_k}^{n_k}(v_{\lambda_k}) \in \partial W_{\lambda_k}$ (non-persistently). Let $k \in \mathbb{N}$ such that $n_k \geq N$ and $d(\lambda_k, \lambda_0) < \frac{\epsilon}{2}$. Let T be a tract above v_{λ_0}, and let $\Phi : T \to \mathbb{H}$ be a Riemann uniformization onto the left half-plane. Let $t_\ell \to -\infty$ be any sequence, and let $\gamma_\ell(\lambda) := \psi_\lambda \circ \Phi^{-1}(-t_\ell)$. For all ℓ large enough, $\gamma_\ell(\lambda_0) = \Phi^{-1}(-t_\ell) \notin S(f_{\lambda_0})$. By Proposition 2.17, there exists $\lambda_* \in \mathbb{B}(\lambda_k, \frac{\epsilon}{2})$ such that $f_{\lambda_*}^{n_k+2}(v_{\lambda_*}) = \gamma_\ell(\lambda_*) = \psi_{\lambda_*} \circ \Phi^{-1}(-t_\ell)$. By Lemma 4.9, for all ℓ large enough, f_{λ_2} has an attracting cycle of period $n_k + 3 > N$, and we are done.

□

5. Characterization of J-stability: Proof of Theorem 1.6

In view of Propositions 2.4 and 2.6, in the proof of Theorem 1.6 we can assume that exceptional maps form a proper analytic subset in the natural family under consideration. Indeed, affine endomorphisms of the complex torus and automorphisms have no singular values; and for rational maps, entire maps, and meromorphic maps, Theorem 1.6 has been proven in [MSS83], [Lyu84], [Lyu83], [EL92] and [ABF21] respectively.

The case of a natural family of finite type self-maps of $\mathbb{C}^*$ with essential singularities at 0 and ∞ is not formally covered by the aforementionned articles. However, the proof given in [EL92] (stated only for finite entire maps) applies verbatim to finite type self-map of $\mathbb{C}^*$. We will therefore only treat the case where the maps f_λ are non exceptional.

Proof of Theorem 1.6.

- (1) $\Leftrightarrow$ (2): This is a particular case of Theorem 1.5, since finite type maps are Ahlfors island maps.
- (2) $\Rightarrow$ (3): This part of the proof follows the same argument as in [MSS83] and [Lyu84]. Assume that the Julia set moves holomorphically over U, and let H_λ be the holomorphic motion respecting the dynamics as above. Then H_λ maps non-attracting periodic points of f_{λ_0} in $J(f_{\lambda_0})$ to non-attracting periodic points of f_λ in $J(f_\lambda)$ of the same period. Let N be the maximal period of attracting cycles for f_{λ_0}. [1] Then for all $\lambda \in U$, cycles of period more than N must be non-attracting, which implies that attracting cycles have period at most N.

[1]Note that we only use a very weak form of Fatou-Shishikura's inequality here, namely that a finite type map f has at most $\operatorname{card} S(f)$ attracting cycles (this is an obvious consequence of the fact that any attracting cycle captures at least one singular value). Epstein has an unpublished proof of a strong version of Fatou-Shishikura's inequality in the setting of finite type maps, which we do not require here.

- $(3) \Rightarrow (1)$: Assume by contraposition that at least one singular value v_λ is active at λ_0, and let $N \in \mathbb{N}$. We must construct a sequence of parameters $\lambda_k \to \lambda_0$ such that f_{λ_k} has an attracting cycle of period at least N.

  We begin with the following remark: let us say that a singular value $v_1(\lambda)$ has a predecessor if there exists another singular value $v_2(\lambda)$ and $n > 0$ such that for all $\lambda \in M$, $f_\lambda^n(v_2(\lambda)) = v_1(\lambda)$. Then clearly $v_1(\lambda)$ is active at λ_0 if and only if its predecessor $v_2(\lambda)$ is also active at λ_0. Moreover, a singular value is its own predecessor if and only if it is persistently periodic; but in that case it cannot be active. Therefore, since there are only finitely many singular values, we may assume without loss of generality that v_λ has no predecessor.

  If v_λ is an asymptotic value, then we are done by Proposition 4.11.

  We therefore assume from now on that v_λ is a critical value with no predecessor. Let c_{λ_0} be a critical point such that $f_{\lambda_0}(c_{\lambda_0}) = v_{\lambda_0}$. If c_{λ_0} is not a singular value, then we are done by Proposition 4.10. Otherwise, c_{λ_0} is both a critical point and a singular value, and then $c_\lambda := \psi_\lambda(c_{\lambda_0})$ is its motion as a critical point and $\varphi_\lambda(c_{\lambda_0})$ is its motion as a singular value. Since by assumption v_λ has no predecessor, the critical point $c_\lambda = \psi_\lambda(c_{\lambda_0})$ cannot always be a singular value for all $\lambda \in M$. This means that $\varphi_\lambda(c_{\lambda_0}) \not\equiv \psi_\lambda(c_{\lambda_0})$. Then the set $H := \{\lambda : \varphi_\lambda(c_{\lambda_0}) = \psi_\lambda(c_{\lambda_0})\}$ is an analytic hypersurface of M, and by Lemma 2.22, we may find $\lambda_1 \notin H$ arbitrarily close to λ_0 such that v_λ is also active at λ_1. Then $c_{\lambda_1} := \psi_{\lambda_1}(c_{\lambda_0}) \notin S(f_{\lambda_1})$, and $f_{\lambda_1}(c_{\lambda_1}) = \varphi_{\lambda_1}(v_{\lambda_0}) = v_{\lambda_1}$. So we can again apply Proposition 4.10 and conclude in this case.

□

Proof of Corollary 1.7. The proof follows the same spirit as in [MSS83]: let $\lambda_0 \in M$, and let $\epsilon > 0$. If all singular values of f_{λ_0} are passive at λ_0, then λ_0 is in the stability locus. Otherwise, at least one attracting singular value is active. If it is a critical value, then by Proposition 4.10 we can find $\lambda_1 \in \mathbb{B}(\lambda_0, \epsilon)$ such that v_{λ_1} is in a super-attracting cycle for f_{λ_1}. In particular, v_λ becomes passive at λ_1. If v_λ is an asymptotic value, then we use Proposition 4.11 and Theorem 4.4 instead to find $\lambda_1 \in \mathbb{B}(\lambda_0, \epsilon)$ such that v_{λ_1} is captured by an attracting cycle (and in particular is passive at λ_1).

Applying this successively to all active singular values, we find $\lambda' \in \mathbb{B}(\lambda_0, k\epsilon)$ (where $k \leq \operatorname{card} S(f_{\lambda_0})$) such that all singular values are passive at λ'. By Theorem 1.6, λ' is then in the stability locus. □

Proof of Corollary 1.8. We may choose without loss of generality λ_0 as the basepoint of our natural family $\{f_\lambda\}_{\lambda \in M}$, that is, we set $f := f_{\lambda_0}$ and we write $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$ for all $\lambda \in M$. The singular value $v(\lambda)$ is by definition $\varphi_\lambda(v(\lambda_0))$.

By Proposition 2.24, there exists a parameter λ_1 arbitrarily close to λ_0 such that $v(\lambda_1)$ is Misiurewicz, i.e. there exists a repelling periodic point $z(\lambda)$ and $n \in \mathbb{N}$ such that

$$f_{\lambda_1}^n(v(\lambda_1)) = z(\lambda_1),$$

and this relation is non-persistent, i.e. $f_\lambda^n(\phi_\lambda(v(\lambda_0))) - z(\lambda) \not\equiv 0$ on the neighborhood of λ_1.

Since singular values move holomorphically with the parameter, $v(\lambda_1)$ is also in the interior of $S(f_{\lambda_1})$.

Moreover, we may choose $z(\lambda)$ so that it is not a Picard exceptional value. Up to passing to a covering (see Lemma 2.16) we can consider x_λ such that $f_\lambda^n(x_\lambda) = z(\lambda)$ such that $x_{\lambda_0} = v(\lambda_0)$. Since v_{λ_0} was in the interior of $S(f_{\lambda_0})$, up to restricting the parameter neighborhood we have that x_λ is a singular value for f_λ. Hence each such parameter has a Misiurewicz relation which involves the singular value $x(\lambda)$. On the other hand, for each λ_* in a neighborhood, $\lambda \mapsto x(\lambda) - \phi_\lambda(x(\lambda_*))$ is not identically zero, hence each such Misiurewicz relation is not persistent and λ_* is in the bifurcation locus, by Lemma 2.21 and Theorem 1.5.

This proves that λ_1 is in the interior of the bifurcation locus, and since λ_1 can be taken arbitrarily close to λ_0, we have that λ_0 is indeed in the closure of the interior of the bifurcation locus. $\square$

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