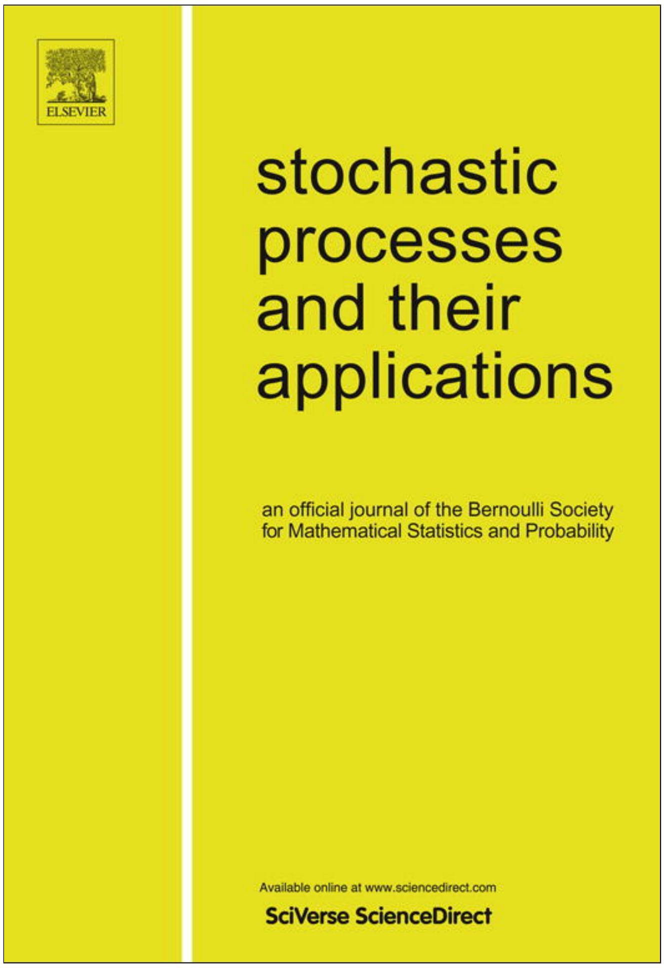
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The forest associated with the record process on a Lévy tree

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Abstract

We perform a pruning procedure on a Lévy tree and instead of throwing away the removed sub-tree, we regraft it on a given branch (not related to the Lévy tree). We prove that the tree constructed by regrafting is distributed as the original Lévy tree, generalizing a result of Addario-Berry, Broutin and Holmgren where only Aldous's tree is considered. As a consequence, we obtain that the "average pruning time" of a leaf is distributed as the height of a leaf picked at random in the Lévy tree.

MSC: 60J80; 60C05

Keywords: Lévy tree; Continuum random tree; Records; Cutting down a tree

1. Introduction

Lévy trees arise as the scaling limits of Galton–Watson trees in the same way as continuous state branching processes (CSBPs) are the scaling limits of Galton–Watson processes (see [16, Chapter 2]). Hence, Lévy trees can be seen as the genealogical trees of some CSBPs, [22]. One can define a random variable $\mathcal{T}$ in the space of real trees (see [19,18,17]) that describes the

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genealogy of a CSBP with branching mechanism ψ of the form:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \left(e^{-\lambda r} - 1 + \lambda r\right) \pi(dr) \quad \text{for } \lambda \geq 0,$$

with $\alpha \geq 0$, $\beta \geq 0$ and π a σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)}(r \wedge r^2)\pi(dr) < +\infty$. We assume that either $\beta > 0$ or $\pi((0, 1)) = +\infty$. In particular, the corresponding CSBP is (sub-) critical as $\psi'(0) = \alpha \geq 0$. In order to use the setting of measured real trees developed in [4], we shall restrict ourselves to compact Lévy trees, that is with branching mechanism satisfying the Grey condition:

$$\int^{+\infty} \frac{dv}{\psi(v)} < +\infty.$$

This condition is equivalent to the compactness of the Lévy tree, and to the a.s. extinction in finite time of the corresponding CSBP.

In [6], a pruning mechanism has been constructed so that the Lévy tree with branching mechanism ψ pruned at rate $q > 0$ is a Lévy tree with branching mechanism ψ_q defined by:

$$\psi_q(\lambda) = \psi(\lambda + q) - \psi(q) \quad \text{for } \lambda \geq 0.$$

This pruning is performed by throwing marks on the tree in a Poissonian manner and by cutting the tree according to these marks, generalizing the fragmentation procedure of the Brownian tree introduced in [8]. This pruning procedure allowed to construct a tree-valued Markov process [2] (see also [9] for an analogous construction in a discrete setting) and to study the record process on Aldous's continuum random tree (CRT) [1] which is related to the number of cuts needed to reduce a Galton–Watson tree.

This problem of cutting down a random tree arises first in [27]: consider a rooted discrete tree with n vertices, pick an edge uniformly at random and remove it together with the sub-tree attached to it and then iterate the procedure on the remaining tree until only the root is left. The question is "How many cuts are needed to isolate the root by this procedure"? Asymptotics in law for this quantity are given in [27] when the tree is a Cayley tree (see also [10,11] in this case where the problem is generalized to the isolation of several leaves and not only the root) and in [24] for conditioned (critical with finite variance) Galton–Watson trees. A.s. convergence has also been obtained in the latter case for a slightly different quantity in [1] using a special pruning procedure that we describe now.

Let $\mathcal{T}$ be a Lévy tree with branching mechanism ψ and $\mathbf{m}^{\mathcal{T}}(dx)$ its "mass measure" supported by the leaves of $\mathcal{T}$. We denote by $\mathbb{P}_r^{\psi}$ the distribution of the Lévy tree corresponding to the CSBP with branching mechanism ψ starting at r and by $\mathbb{N}^{\psi}$ the corresponding excursion measure also called canonical measure (in particular, $\mathbb{P}_r^{\psi}$ can be seen as the distribution of a "forest" of Lévy trees given by a Poisson point measure with intensity $r\mathbb{N}^{\psi}$). The branching points of the Lévy tree are either binary or of infinite degree (see [17, Theorem 4.6]) and to each infinite degree branching point x, one can associate a size Δ_x which measures in some sense the number of sub-trees attached to it (see (6) in Section 2.5). We then consider a measure $\mu^{\mathcal{T}}$ on $\mathcal{T}$ defined by:

$$\mu^{\mathcal{T}}(dy) = 2\beta\ell^{\mathcal{T}}(dy) + \sum_{x \in \mathrm{Br}_\infty(\mathcal{T})} \Delta_x \delta_x(dy),$$

where $\ell^{\mathcal{T}}$ is the length measure on the skeleton of the tree, $\mathrm{Br}_\infty(\mathcal{T})$ is the set of branching points of infinite degree and δ_x is the Dirac measure at point x. Aldous's CRT corresponds to the distribution of $\mathcal{T}$ under $\mathbb{N}^\psi$, with $\psi(\lambda) = \frac{1}{2}\lambda^2$, and conditionally on $\mathbf{m}^{\mathcal{T}}(\mathcal{T}) = 1$. In this case $\mathrm{Br}_\infty(\mathcal{T})$ is empty and thus $\mu^{\mathcal{T}}(dy) = \ell^{\mathcal{T}}(dy)$.

Then we consider, conditionally given $\mathcal{T}$, a Poisson point process $M^{\mathcal{T}}(d\theta, dy)$ of marks on the tree with intensity

$$\mathbf{1}_{[0,+\infty)}(\theta) d\theta\, \mu^{\mathcal{T}}(dy).$$

Parameter y indicates the location of the mark whereas θ represents the time at which it appears. For every $x \in \mathcal{T}$, we set

$\theta(x)$ *the first time* θ *at which a mark appears between* x *and the root.*

We consider Θ the average of these first cutting times over the Lévy tree:

$$\Theta = \int_{\mathcal{T}} \theta(x) \mathbf{m}^{\mathcal{T}}(dx).$$

It has been proven in [1] (Theorem 6.1 and Corollary 5.3 with $\psi(u) = u^2/2$) in the framework of Aldous's CRT, that if we denote by X_n the number of cuts needed to isolate the root in the sub-tree spanned by n leaves randomly chosen, then a.s. $\lim_{n\to+\infty} X_n/L_n = \Theta$, with $L_n \sim \sqrt{2n}$ the total length of the sub-tree. Moreover, the law of Θ in that case is a Rayleigh distribution (i.e. with density $x\mathrm{e}^{-x^2/2}\mathbf{1}_{\{x\geq 0\}}$). The distribution of Θ is also the law of the height of a leaf picked at random in Aldous's tree. This surprising relationship is explained by Addario-Berry, Broutin and Holmgren in [7, Theorem 10]. The authors consider a branch with length Θ, and when a mark appears, the tree is cut and the sub-tree which does not contain the root is removed and grafted on this branch (the grafting position is described using some local time). Then the new tree obtained by this grafting procedure is again distributed as Aldous's tree.

The goal of this paper is to generalize this result to general Lévy trees. We consider a Lévy tree $\mathcal{T}$ under $\mathbb{N}^\psi$ and we perform the pruning procedure described above. When a mark appears, we remove the sub-tree attached to this mark and keep the sub-tree containing the root. We denote by $\mathcal{T}_q$ the resulting tree at time q i.e. the set of points of the initial tree $\mathcal{T}$ which have no marks between them and the root at time q:

$$\mathcal{T}_q = \{x \in \mathcal{T}; \theta(x) \geq q\}.$$

According to [6, Theorem 1.1], $\mathcal{T}_q$ is a Lévy tree with branching mechanism ψ_q. We consider Θ_q the average of the records shifted by q over the Lévy tree $\mathcal{T}_q$:

$$\Theta_q = \int_{\mathcal{T}_q} (\theta(x) - q)\, \mathbf{m}^{\mathcal{T}}(dx).$$

Remark that a.s. $\mathcal{T}_q \subset \mathcal{T}$ and hence $\Theta_q \leq \Theta$.

We define an equivalence relation on the tree $\mathcal{T}$: $x \sim y$ if the function θ remains constant on the path between x and y. We consider the equivalence classes $(\mathcal{T}^i, i \in I^R)$ and denote for each $i \in I^R$ by θ_i the common value of the function θ. In the pruning procedure described above, the tree $\mathcal{T}^i$ corresponds to the sub-tree which is removed at time θ_i and it is distributed according to $\mathbb{N}^{\psi_{\theta_i}}$. Then we consider a branch B^R of length Θ rooted at some end point, say $\emptyset$. The sub-tree $\mathcal{T}^i$ is grafted on B^R at distance Θ_{θ_i} from the root; see Fig. 1. Let $\mathcal{T}^R$ denote this tree obtained by regrafting. Our main result, see Theorem 3.1, relies on Laplace transform computations and can be stated as follows.

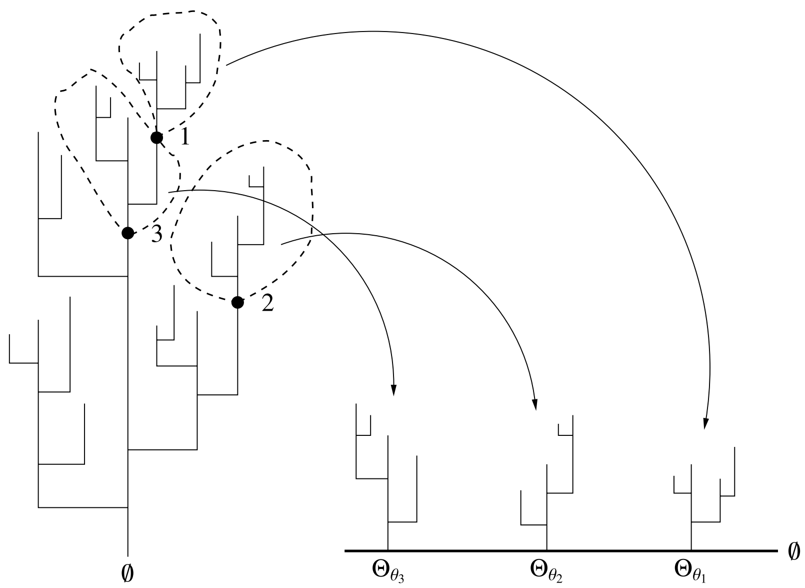

Fig. 1. Pruning of a Lévy tree (left) and tree $\mathcal{T}^R$ obtained by regrafting on a branch (right). The marks are numbered according to their order of appearance.

Theorem. *Assume that the Grey condition holds. Under $\mathbb{N}^\psi$, $(B^R, \mathcal{T}^R)$ is distributed as $(B, \mathcal{T})$ where B is a branch from the root $\emptyset$ to a leaf chosen at random on $\mathcal{T}$ according to the mass measure $\mathbf{m}^\mathcal{T}$.*

In particular, this theorem implies the following corollary.

Corollary. *Under $\mathbb{N}^\psi[d\mathcal{T}]$, Θ is distributed as the height H of a leaf of the Lévy tree chosen at random according to the mass measure $\mathbf{m}^\mathcal{T}$.*

A probabilistic interpretation of those results for the Brownian CRT is provided in [7] using a path transformation on Brownian bridge or in [10] using a fragmentation tree. We do not know if such an approach is valid in the present general framework.

Using the Bismut decomposition of Lévy trees (see Fig. 2), we recover and extend to general Lévy trees Proposition 8.2 from [2] on the asymptotics of the masses of $(\mathcal{T}^i, i \in I^R)$. Set $\sigma = \mathbf{m}^\mathcal{T}(\mathcal{T})$ and $\sigma^i = \mathbf{m}^\mathcal{T}(\mathcal{T}^i)$ for $i \in I^R$.

Corollary. *Assume that the Grey condition holds. $\mathbb{N}^\psi$-a.e., we have:*

$$\lim_{\varepsilon\to 0} \frac{1}{\mathbb{N}^\psi[\sigma > \varepsilon]} \sum_{i\in I^R} \mathbf{1}_{\{\sigma^i \geq \varepsilon\}} = \Theta.$$

Similar results hold for the convergence of $\frac{1}{\mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma\leq\varepsilon\}}]} \sum_{i\in I^R} \sigma^i \mathbf{1}_{\{\sigma^i\leq\varepsilon\}}$ to Θ; see Corollary 3.2.

The above theorem states that the point process with atoms $(\Theta_{\theta_i}, \mathcal{T}^i)$, $i \in I^R$ is distributed as the point process that appears in the Bismut decomposition of a Lévy tree. This may seem quite surprising. Indeed, if $\theta_i \leq \theta_j$, then $\mathcal{T}^i$ is stochastically greater than $\mathcal{T}^j$ (as a tree distributed according to $\mathbb{N}^{\psi_q}$ can be obtained from a tree distributed according to $\mathbb{N}^{\psi_{q'}}$ for $q \geq q'$ by pruning). Consequently, the trees that are grafted on B^R are in some sense smaller and smaller whereas the trees in the Bismut decomposition have the same law. However, the intensity of the

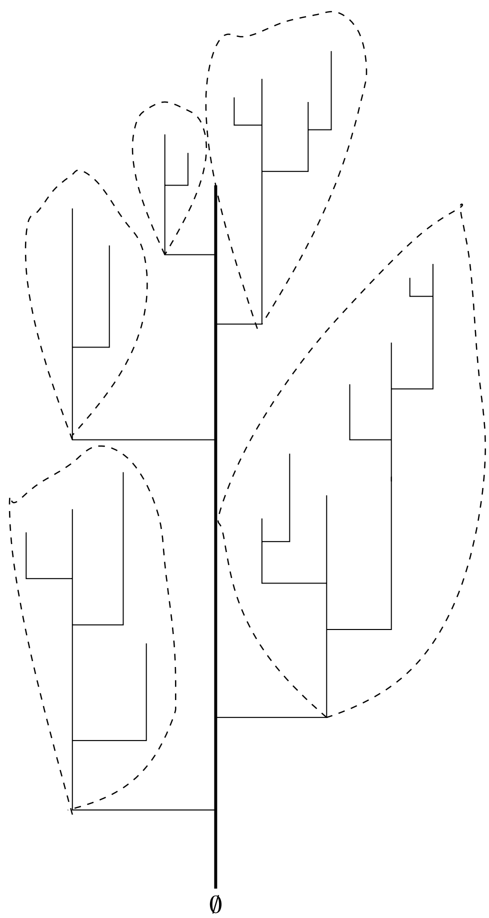

Fig. 2. Bismut decomposition of a Lévy tree.

grafting is not uniform in the first case (contrary to the Bismut decomposition) and depends on the size of the trees grafted before, which gives at the end the identity in distribution.

In the present work, we ignore the marks that fall on the sub-trees once they have been removed. However, we could use them to iterate our construction on each sub-tree $(\mathcal{T}^i, i \in I^R)$ and so on, in order to generalize to general Lévy trees the result obtained for Aldous's CRT by Bertoin and Miermont [11].

In view of the present work, we conjecture that similar results to [24] hold for infinite variance offspring distribution. Let X_n denote the number of cuts needed to isolate the root by pruning at edges a Galton–Watson tree conditioned to have n vertices. We also consider the pruning at vertices inspired by Abraham et al. [3], which is the discrete analogue of the continuous pruning: pick an edge uniformly at random and remove *the vertex from which the edge comes* together with the sub-tree attached to this vertex. Let $\tilde{X}_n$ be the number of cuts until the root is removed by this procedure for a Galton–Watson tree conditioned to have n vertices. According to [24], the number of cuts needed to remove the root for the pruning at vertices (that is $\tilde{X}_n$) or to isolate the root for the pruning at edges (that is X_n) are asymptotically equivalent for finite variance offspring distribution. However, we expect a different behaviour in the infinite variance case. Consider a critical Galton–Watson tree with offspring distribution in the domain of attraction of a stable law of index $\gamma \in (1, 2]$. According to [15] or [25], the (contour process of the) Galton–Watson tree conditioned to have total progeny n, properly rescaled, converges in distribution to (the contour process of) a Lévy tree under $\mathbb{N}^\psi[\,\cdot\,|\sigma = 1]$, with $\psi(\lambda) = c_0\lambda^\gamma$ for some $c_0 > 0$.

Conjecture. *Let L_n denote the length of the rescaled Galton–Watson tree conditioned to have total progeny n. We conjecture that:*

$$\frac{\tilde{X}_n}{L_n} \xrightarrow[n\to+\infty]{(d)} Z,$$

for some random variable Z distributed as the height of a leaf chosen at random according to the mass measure under $\mathbb{N}^\psi[\,\cdot\,|\sigma = 1]$.

Set $a = (\gamma - 1)/\gamma$. Using the Laplace transform (see Theorem 2.1), we get that the height H of a leaf randomly chosen in the Lévy tree is distributed under $\mathbb{N}^\psi$ as $\sigma^a Z$, with Z and σ independent and the distribution of Z is characterized for $n \in \mathbb{N}$ by:

$$\mathbb{E}\left[Z^n\right] = \frac{1}{c_0^{n/\gamma}\gamma^n} \frac{\Gamma(a)\Gamma(n+1)}{\Gamma(a(n+1))}.$$

In the particular case of Aldous's CRT, $\gamma = 2$ and $c_0 = 1/2$, we recover, using the duplication formula of the gamma function, that Z (and thus H under $\mathbb{N}^\psi[\,\cdot\,|\sigma = 1]$) has Rayleigh distribution.

The paper is organized as follows. We collect results on Lévy trees in Section 2, with the Bismut decomposition in Section 2.7 and the pruning procedure in Section 2.8. The main result is then precisely stated in Section 3 and proved in Section 4.

2. Lévy trees and the forest obtained by pruning

2.1. Notations

Let (E, d) be a metric Polish space. For $x \in E$, δ_x denotes the Dirac measure at point x. For μ a Borel measure on E and f a non-negative measurable function, we set $\langle \mu, f\rangle = \int f(x)\,\mu(dx) = \mu(f)$.

2.2. Real trees

We refer the reader to [12,14,28] for a general presentation of $\mathbb{R}$-trees and to [18] or [21] for their applications in the field of random real trees. Informally, real trees are metric spaces without loops, locally isometric to the real line. More precisely, a metric space (T, d) is a real tree if the following properties are satisfied.

(1) For every $s, t \in T$, there is a unique isometric map $f_{s,t}$ from $[0, d(s, t)]$ to T such that $f_{s,t}(0) = s$ and $f_{s,t}(d(s, t)) = t$.
(2) For every $s, t \in T$, if q is a continuous injective map from $[0, 1]$ to T such that $q(0) = s$ and $q(1) = t$, then $q([0, 1]) = f_{s,t}([0, d(s, t)])$.

If $s, t \in T$, we will denote by $[\![s, t]\!]$ the range of the isometric map $f_{s,t}$ described above. We will also write $[\![s, t[\![$ for the set $[\![s, t]\!] \setminus \{t\}$.

We say that $(T, d, \emptyset)$ is a rooted real tree with root $\emptyset$ if (T, d) is a real tree and $\emptyset \in T$ is a distinguished vertex.

Let $(T, d, \emptyset)$ be a rooted real tree. If $x \in T$, the degree of x, $n(x)$, is the number of connected components of $T \setminus \{x\}$. We shall consider the set of leaves $\mathrm{Lf}(T) = \{x \in T \setminus \{\emptyset\},\ n(x) = 1\}$, the set of branching points $\mathrm{Br}(T) = \{x \in T,\ n(x) \geq 3\}$ and the set of infinite branching points

$\mathrm{Br}_\infty(T) = \{x \in T,\ n(x) = \infty\}$. The skeleton of T is the set of points in the tree that are not leaves: $\mathrm{Sk}(T) = T \setminus \mathrm{Lf}(T)$. The trace of the Borel σ-field of T restricted to $\mathrm{Sk}(T)$ is generated by the sets $[\![s, s']\!]$; $s, s' \in \mathrm{Sk}(T)$. Hence, one defines uniquely a σ-finite Borel measure ℓ^T on T, called length measure of T, such that:

$$\ell^T(\mathrm{Lf}(T)) = 0 \quad \text{and} \quad \ell^T([\![s, s']\!]) = d(s, s').$$

For every $x \in T$, $[\![\emptyset, x]\!]$ is interpreted as the ancestral line of vertex x in the tree. We define a partial order on T by setting $x \preccurlyeq y$ (x is an ancestor of y) if $x \in [\![\emptyset, y]\!]$. If $x, y \in T$, there exists a unique $z \in T$, called the Most Recent Common Ancestor (MRCA) of x and y, such that $[\![\emptyset, x]\!] \cap [\![\emptyset, y]\!] = [\![\emptyset, z]\!]$. We write $z = x \wedge y$.

2.3. Measured rooted real trees

We call a w-tree a weighted rooted real tree, i.e. a quadruplet $(T, d, \emptyset, \mathbf{m})$ where $(T, d, \emptyset)$ is a locally compact rooted real tree and $\mathbf{m}$ is a locally finite measure on T. Sometimes, we will write $(T, d^T, \emptyset^T, \mathbf{m}^T)$ for $(T, d, \emptyset, \mathbf{m})$ to stress the dependence in T, or simply T when there is no confusion. We denote by $\mathfrak{T}$ the set of w-trees.

In order to define a tractable distance on w-trees, we need an equivalence relation between two w-trees, i.e. we identify two w-trees $(T, d^T, \emptyset^T, \mathbf{m}^T)$ and $(T', d^{T'}, \emptyset^{T'}, \mathbf{m}^{T'})$ if there exists an isometric function which maps T onto T', which sends $\emptyset^T$ onto $\emptyset^{T'}$ and which transports measure $\mathbf{m}^T$ on measure $\mathbf{m}^{T'}$. We will denote by $\mathbb{T}$ the set of measure-preserving and root-preserving isometry classes of w-trees. One can define a topology on $\mathbb{T}$ such that $\mathbb{T}$ is a Polish space; see for example [20,23,5].

Let $T, T' \in \mathfrak{T}$ be w-trees that belong to the same equivalence class. Let φ be a measure-preserving–root-preserving isometry that maps T onto T'. A $\mathfrak{T}$-valued function F of the form $F(T, (x_i, i \in I))$ where $(x_i, i \in I)$ is a family of points of T is said to be $\mathbb{T}$-compatible if $F(T', (\varphi(x_i), i \in I))$ belongs to the same equivalence class as $F(T, (x_i, i \in I))$.

Let $T \in \mathfrak{T}$. For $x \in T$, we set $h(x) = d(\emptyset, x)$ the height of x and

$$H_{\max}(T) = \sup_{x \in T} h(x) \tag{1}$$

the height of the tree (possibly infinite). Remark that for two w-trees in the same equivalence class, the heights are the same; hence $H_{\max}(T)$ is well-defined for $T \in \mathbb{T}$.

For $a \geq 0$, we set:

$$T(a) = \{x \in T, d(\emptyset, x) = a\} \quad \text{and} \quad \pi_a(T) = \{x \in T, d(\emptyset, x) \leq a\},$$

the restriction of the tree T at level a and the truncated tree T up to level a. We consider $\pi_a(T)$ with the induced distance, the root $\emptyset$ and the mass measure $\mathbf{m}^{\pi_a(T)}$ which is the restriction of $\mathbf{m}^T$ to $\pi_a(T)$, to get a w-tree. Let us remark that the map π_a is $\mathbb{T}$-compatible. We denote by $(T^{i,\circ}, i \in I)$ the connected components of $T \setminus \pi_a(T)$. Let $\emptyset_i$ be the MRCA of all the points of $T^{i,\circ}$. We consider the real tree $T^i = T^{i,\circ} \cup \{\emptyset_i\}$ rooted at point $\emptyset_i$ with mass measure $\mathbf{m}^{T^i}$ defined as the restriction of $\mathbf{m}^T$ to T^i. We will consider the point measure on $T \times \mathbb{T}$:

$$\mathcal{N}_a^T = \sum_{i \in I} \delta_{(\emptyset_i, T^i)}.$$

2.4. Grafting procedure

We will define in this section a procedure by which we add (graft) w-trees on an existing w-tree. More precisely, let $T \in \mathfrak{T}$ and let $((T_i, x_i), i \in I)$ be a finite or countable family of elements of $\mathbb{T} \times T$. We define the real tree obtained by grafting the trees T_i on T at point x_i. We set $\tilde{T} = T \sqcup \left(\bigsqcup_{i\in I} T_i \setminus \{\emptyset^{T_i}\}\right)$ where the symbol $\sqcup$ means that we choose for the sets $(T_i)_{i\in I}$ representatives of isometry classes in $\mathbb{T}$ which are disjoint subsets of some common set and that we perform the disjoint union of all these sets. We set $\emptyset^{\tilde{T}} = \emptyset^T$. The set $\tilde{T}$ is endowed with the following metric $d^{\tilde{T}}$: if $s, t \in \tilde{T}$,

$$d^{\tilde{T}}(s,t) = \begin{cases} d^T(s,t) & \text{if } s,t \in T, \\ d^T(s,x_i) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } s \in T, t \in T_i \setminus \{\emptyset^{T_i}\}, \\ d^{T_i}(s,t) & \text{if } s,t \in T_i \setminus \{\emptyset^{T_i}\}, \\ d^T(x_i,x_j) + d^{T_j}(\emptyset^{T_j}, s) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } i \neq j \text{ and } s \in T_j \setminus \{\emptyset^{T_j}\}, \\ & \quad t \in T_i \setminus \{\emptyset^{T_i}\}. \end{cases}$$

We define the mass measure on $\tilde{T}$ by:

$$\mathbf{m}^{\tilde{T}} = \mathbf{m}^T + \sum_{i\in I} \left(\mathbf{1}_{T_i \setminus \{\emptyset^{T_i}\}} \mathbf{m}^{T_i} + \mathbf{m}^{T_i}(\{\emptyset^{T_i}\}) \delta_{x_i} \right),$$

where δ_x is the Dirac mass at point x. We will use the following notation:

$$(\tilde{T}, d^{\tilde{T}}, \emptyset^{\tilde{T}}, \mathbf{m}^{\tilde{T}}) = T \circledast_{i\in I} (T_i, x_i). \tag{2}$$

It is clear that the metric space $(\tilde{T}, d^{\tilde{T}}, \emptyset^{\tilde{T}})$ is still a rooted complete real tree. Notice that it is not always true that $\tilde{T}$ remains locally compact or that $\mathbf{m}^{\tilde{T}}$ defines a locally finite measure on $\tilde{T}$. For instance if we consider the grafting $\{\emptyset\} \circledast_{n\in\mathbb{N}} (T, \emptyset)$ where T is a non-trivial tree (i.e. we graft the same tree an infinite number of times on a single point), then the resulting tree is not locally compact.

It is easy to check that the function defined by $F(T, (x_i, i \in I)) = T \circledast_{i\in I} (T_i, x_i)$ if $T \circledast_{i\in I} (T_i, x_i)$ belongs to $\mathfrak{T}$ and $F(T, (x_i, i \in I)) = T$ otherwise is $\mathbb{T}$-compatible. That is the grafting procedure, when the resulting tree belongs to $\mathfrak{T}$, is $\mathbb{T}$-compatible.

2.5. Excursion measure of a Lévy tree

Let ψ be a critical or sub-critical branching mechanism defined by:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \left(\mathrm{e}^{-\lambda r} - 1 + \lambda r \right) \pi(dr) \tag{3}$$

with $\alpha \geq 0$, $\beta \geq 0$ and π is a σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)} (r \wedge r^2) \pi(dr) < +\infty$ and $\langle \pi, 1 \rangle = +\infty$ if $\beta = 0$. We also assume the Grey condition:

$$\int^{+\infty} \frac{d\lambda}{\psi(\lambda)} < +\infty. \tag{4}$$

The Grey condition is equivalent to the a.s. finiteness of the extinction time of the CSBP. This assumption is used to ensure that the corresponding Lévy tree is compact. Let v be the unique non-negative solution of the equation:

$$\forall a > 0, \quad \int_{v(a)}^{+\infty} \frac{d\lambda}{\psi(\lambda)} = a.$$

We gather here results from [17, Theorems 4.2, 4.3 and 4.6, 4.7]. Remarks of pages 575 and 578 of [17] state that the local time ℓ^a is a function of the tree (see the third property below) and hence can be defined on $\mathbb{T}$.

Using the coding of compact real trees by height functions, we can define a σ-finite measure $\mathbb{N}^\psi[d\mathcal{T}]$ on $\mathbb{T}$, or excursion measure of the Lévy tree, with the following properties.

(i) *Height.* Recall Definition (1) of the height $H_{\max}(\mathcal{T})$ of a tree. For all $a > 0$,

$$\mathbb{N}^\psi[H_{\max}(\mathcal{T}) > a] = v(a).$$

(ii) *Mass measure.* The mass measure $\mathbf{m}^{\mathcal{T}}$ is supported by $\mathrm{Lf}(\mathcal{T})$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.

(iii) *Local time.* There exists a process $(\ell^a, a \geq 0)$ with values on finite measures on $\mathcal{T}$, which is càdlàg for the weak topology on finite measures on $\mathcal{T}$ and such that $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:

$$\mathbf{m}^{\mathcal{T}}(dx) = \int_0^\infty \ell^a(dx)\, da, \tag{5}$$

$\ell^0 = 0$, $\inf\{a > 0;\ \ell^a = 0\} = \sup\{a \geq 0;\ \ell^a \neq 0\} = H_{\max}(\mathcal{T})$ and for every fixed $a \geq 0$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:

- The measure ℓ^a is supported on $\mathcal{T}(a)$.
- We have for every bounded continuous function ϕ on $\mathcal{T}$:

$$\begin{aligned}\langle \ell^a, \phi\rangle &= \liminf_{\epsilon\downarrow 0} \frac{1}{v(\epsilon)} \int \phi(x) \mathbf{1}_{\{H_{\max}(\mathcal{T}')\geq\epsilon\}} \mathcal{N}_a^{\mathcal{T}}(dx, d\mathcal{T}')\\ &= \liminf_{\epsilon\downarrow 0} \frac{1}{v(\epsilon)} \int \phi(x) \mathbf{1}_{\{H_{\max}(\mathcal{T}')\geq\epsilon\}} \mathcal{N}_{a-\epsilon}^{\mathcal{T}}(dx, d\mathcal{T}'), \quad \text{if } a > 0.\end{aligned}$$

Moreover, the above lim inf are true limits $\mathbb{N}^\psi[d\mathcal{T}]$-a.s.

Under $\mathbb{N}^\psi$, the real valued process $(\langle \ell^a, 1\rangle, a \geq 0)$ is distributed as a CSBP with branching mechanism ψ under its canonical measure.

(iv) *Branching property.* For every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^{\mathcal{T}}(dx, d\mathcal{T}')$ under $\mathbb{N}^\psi[d\mathcal{T} | H_{\max}(\mathcal{T}) > a]$, given $\pi_a(\mathcal{T})$, is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(dx)\mathbb{N}^\psi[d\mathcal{T}']$.

(v) *Branching points.*

- $\mathbb{N}^\psi[d\mathcal{T}]$-a.e., the branching points of $\mathcal{T}$ are of degree 3 or $+\infty$.
- The set of binary branching points (i.e. of degree 3) is empty $\mathbb{N}^\psi$ a.e if $\beta = 0$ and is a countable dense subset of $\mathcal{T}$ if $\beta > 0$.
- The set $\mathrm{Br}_\infty(\mathcal{T})$ of infinite branching points is nonempty with $\mathbb{N}^\psi$-positive measure if and only if $\pi \neq 0$. If $\langle \pi, 1\rangle = +\infty$, the set $\mathrm{Br}_\infty(\mathcal{T})$ is $\mathbb{N}^\psi$-a.e. a countable dense subset of $\mathcal{T}$.

(vi) *Mass of the nodes.* The set $\{d(\emptyset, x),\ x \in \mathrm{Br}_\infty(\mathcal{T})\}$ coincides $\mathbb{N}^\psi$-a.e. with the set of discontinuity times of the mapping $a \mapsto \ell^a$. Moreover, $\mathbb{N}^\psi$-a.e., for every such discontinuity time b, there is a unique $x_b \in \mathrm{Br}_\infty(\mathcal{T}) \cap \mathcal{T}(b)$ and $\Delta_b > 0$, such that:

$$\ell^b = \ell^{b-} + \Delta_b \delta_{x_b},$$

where $\Delta_b > 0$ is called the mass of the node x_b. Furthermore Δ_b can be obtained by the approximation:

$$\Delta_b = \liminf_{\epsilon\to 0} \frac{1}{v(\epsilon)} n(x_b, \epsilon), \tag{6}$$

where $n(x_b, \epsilon) = \int \mathbf{1}_{\{x=x_b\}} \mathbf{1}_{\{H_{max}(\mathcal{T}')>\epsilon\}} \mathcal{N}_b^{\mathcal{T}}(dx, d\mathcal{T}')$ is the number of sub-trees originating from x_b with height larger than ϵ. Moreover, the above lim inf is a true limit $\mathbb{N}^\psi[d\mathcal{T}]$-a.s.

In order to stress the dependence in $\mathcal{T}$, we may write $\ell^{a,\mathcal{T}}$ for ℓ^a.

We set $\sigma^{\mathcal{T}}$ or simply σ when there is no confusion, the total mass of the mass measure on $\mathcal{T}$:

$$\sigma = \mathbf{m}^{\mathcal{T}}(\mathcal{T}). \tag{7}$$

In particular, as σ is distributed as the total mass of a CSBP under its canonical measure, we have that $\mathbb{N}^\psi$-a.s. $\sigma > 0$ and for $q > 0$ (see for instance [26, Corollary 10.9] for the first equality, the others being obtained by differentiation):

$$\mathbb{N}^\psi\left[1 - \mathrm{e}^{-\psi(q)\sigma}\right] = q, \qquad \mathbb{N}^\psi\left[\sigma \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{1}{\psi'(q)} \quad \text{and}$$

$$\mathbb{N}^\psi\left[\sigma^2 \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{\psi''(q)}{\psi'(q)^3}. \tag{8}$$

The last two equations hold for $q = 0$ if $\psi'(0) > 0$.

2.6. *Other measures on* $\mathbb{T}$

For each $r > 0$, we define a probability measure $\mathbb{P}_r^\psi$ on $\mathbb{T}$ as follows. Let $r > 0$ and $\sum_{k\in\mathcal{K}} \delta_{\mathcal{T}^k}$ be a Poisson point measure on $\mathbb{T}$ with intensity $r\mathbb{N}^\psi$. Consider $\{\emptyset\}$ as the trivial w-tree reduced to the root with null mass measure. Define $\mathcal{T} = \{\emptyset\} \circledast_{k\in\mathcal{K}} (\mathcal{T}^k, \emptyset)$. Using Property (i) as well as (8), one easily gets that for every $\varepsilon > 0$ there is only a finite number of trees $\mathcal{T}^k$ with height larger than ε. As each tree $\mathcal{T}^k$ is compact, we deduce that $\mathcal{T}$ is a compact w-tree, and hence belongs to $\mathbb{T}$. We denote by $\mathbb{P}_r^\psi$ its distribution. Its corresponding local time is defined by $\ell^a = \sum_{k\in\mathcal{K}} \ell^{a,\mathcal{T}^k}$ and its total mass is defined by $\sigma = \sum_{k\in\mathcal{K}} \sigma^{\mathcal{T}^k}$. Under $\mathbb{P}_r^\psi$, the real valued process $(\langle \ell^a, 1\rangle, a \geq 0)$ is distributed as a CSBP with branching mechanism ψ with initial value r.

We consider the following measure on $\mathbb{T}$:

$$\mathbf{N}^\psi[d\mathcal{T}] = 2\beta \mathbb{N}^\psi[d\mathcal{T}] + \int_0^{+\infty} r\pi(dr)\, \mathbb{P}_r^\psi(d\mathcal{T}) \tag{9}$$

which appears as the grafting intensity in the tree-valued Markov process of [4]. From (8) and (9), elementary computations yield for $q > 0$:

$$\mathbf{N}^\psi\left[1 - \mathrm{e}^{-\psi(q)\sigma}\right] = \psi'(q) - \psi'(0), \tag{10}$$

as well as

$$\mathbf{N}^\psi\left[\sigma \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{\psi''(q)}{\psi'(q)} \quad \text{and} \quad \mathbf{N}^\psi\left[\sigma^2 \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{1}{\psi'(q)} \partial_q \left(\frac{-\psi''(q)}{\psi'(q)}\right). \tag{11}$$

The last two equalities also hold for $q = 0$ if $\psi'(0) > 0$.

2.7. *Bismut decomposition of a Lévy tree*

We first present a decomposition of $T \in \mathfrak{T}$ according to a given vertex $x \in T$. We denote by $(T^{j,\circ}, j \in J_x)$ the connected components of $T \setminus [\![\emptyset, x]\!]$. For every $j \in J_x$, let x_j be the MRCA of

$T^{j,\circ}$ and consider $T^j = T^{j,\circ} \cup \{x_j\}$ as an element of $\mathbb{T}$ with mass measure the mass measure of T restricted to $T^{j,\circ}$. In order to graft together all the sub-trees with the same MRCA, we consider the following equivalence relation on J_x:

$$j \sim j' \iff x_j = x_{j'}.$$

Let I_x^B be the set of equivalence classes. For $[i] \in I_x^B$, we set $x_{[i]}$ for the common value of x_j with $j \in [i]$. We consider $\{x_{[i]}\}$ as an element of $\mathbb{T}$ with 0 mass measure. For $[i] \in I_x^B$, we consider the following element of $\mathbb{T}$ defined by:

$$T^{B,[i]} = \{x_{[i]}\} \circledast_{j\in[i]} (T^j, x_{[i]}).$$

Let $h_{[i]} = d(\emptyset, x_{[i]})$. We consider the random point measure $\mathcal{M}_x^T$ on $\mathbb{R}_+ \times \mathbb{T}$ defined by:

$$\mathcal{M}_x^T = \sum_{[i]\in I_x^B} \delta_{(h_{[i]}, T^{B,[i]})}.$$

Under $\mathbb{N}^\psi$, conditionally on $\mathcal{T}$, let U be a $\mathcal{T}$-valued random variable, with distribution $\sigma^{-1}\,\mathbf{m}^{\mathcal{T}}$. In other words, conditionally on $\mathcal{T}$, U represents a leaf chosen "uniformly" at random according to the mass measure $\mathbf{m}^{\mathcal{T}}$. We define under $\mathbb{N}^\psi$ a non-negative random variable and a random point measure on $\mathbb{R}_+ \times \mathbb{T}$ as follows:

$$H = d^{\mathcal{T}}(\emptyset^{\mathcal{T}}, U) \quad \text{and} \quad \mathcal{Z}^B = \mathcal{M}_U^{\mathcal{T}}. \tag{12}$$

Let us remark that the distribution of $(H, \mathcal{Z}^B)$ does not depend on the choice of the representative in the equivalence class and thus this random variable is well defined under $\mathbb{N}^\psi$.

By construction, for every non-negative measurable function Φ on $\mathbb{R}_+ \times \mathbb{T}$ and for every $\lambda \geq 0$, $\rho \geq 0$, we have:

$$\mathbb{N}^\psi\left[\sigma \mathrm{e}^{-\lambda\sigma-\rho H-\langle \mathcal{Z}^B, \Phi\rangle}\right] = \mathbb{N}^\psi\left[\int_{\mathcal{T}} \mathbf{m}^{\mathcal{T}}(dx)\,\mathrm{e}^{-\lambda\sigma-\rho h(x)-\langle \mathcal{M}_x^{\mathcal{T}}, \Phi\rangle}\right].$$

As a direct consequence of Theorem 4.5 of [17], we get the following result.

Theorem 2.1. *For every non-negative measurable function Φ on $\mathbb{R}_+ \times \mathbb{T}$ and for every $\lambda \geq 0$, $\rho \geq 0$, we have:*

$$\mathbb{N}^\psi\left[\sigma \mathrm{e}^{-\lambda\sigma-\rho H-\langle \mathcal{Z}^B, \Phi\rangle}\right] = \int_0^{+\infty} da\mathrm{e}^{-\rho a} \exp\left(-\int_0^a g(\lambda, u)du\right),$$

where

$$g(\lambda, u) = \psi'(0) + \mathbf{N}^\psi\left[1 - \mathrm{e}^{-\lambda\sigma-\Phi(u,\mathcal{T})}\right]. \tag{13}$$

In other words, under $\mathbb{N}^\psi[\sigma, d\mathcal{T}]$, if we choose a leaf U uniformly (i.e. according to the normalized mass measure $\mathbf{m}^{\mathcal{T}}$), the height H of this leaf is distributed according to the density $da\mathrm{e}^{-\psi'(0)a}$ and, conditionally on H, the point measure $\mathcal{Z}^B$ is a Poisson point process on $[0, H]$ with intensity $\mathbf{N}^\psi[d\mathcal{T}]$.

2.8. Pruning a Lévy tree

A general pruning of a Lévy tree has been defined in [6]. We use a special case of this pruning depending on a one-dimensional parameter θ used first in [29] to define a fragmentation process of the tree.

More precisely, under $\mathbb{N}^\psi[d\mathcal{T}]$, we consider a mark process $M^{\mathcal{T}}(d\theta, dy)$ on the tree which is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{T}$ with intensity:

$$\mathbf{1}_{[0,+\infty)}(\theta)d\theta \left(2\beta\ell^{\mathcal{T}}(dy) + \sum_{x\in \mathrm{Br}_\infty(\mathcal{T})} \Delta_x \delta_x(dy)\right).$$

The atoms $(\theta_i, y_i)_{i\in I}$ of this measure can be seen as marks that arrive on the tree, y_i being the location of the mark and θ_i the "time" at which it appears. There are two kinds of marks: some are "uniformly" distributed on the skeleton of the tree (they correspond to the term $2\beta\ell^{\mathcal{T}}$ in the intensity) whereas the others are located on the infinite branching points of the tree, an infinite branching point y being first marked after an exponential time with parameter Δ_y.

For every $x \in \mathcal{T}$, we set:

$$\theta(x) = \inf\{\theta > 0, M^{\mathcal{T}}([0, \theta] \times [\![\emptyset, x]\!]) > 0\}.$$

The process $(\theta(x), x \in \mathcal{T})$ is called the record process on the tree $\mathcal{T}$ as defined in [1]. This corresponds to the first time at which a mark arrives on $[\![\emptyset, x]\!]$. Using this record process, we define the pruned tree at time q as:

$$\mathcal{T}_q = \{x \in \mathcal{T},\ \theta(x) \geq q\}$$

with the induced metric, root $\emptyset$ and mass measure the restriction of the mass measure $\mathbf{m}^{\mathcal{T}}$. If one cuts the tree $\mathcal{T}$ at time θ_i at point y_i, then $\mathcal{T}_q$ is the sub-tree of $\mathcal{T}$ containing the root at time q. Here again, the definition of $\mathcal{T}_q$ is $\mathbb{T}$-compatible.

Proposition 2.2 ([6, Theorem 1.1]). *For $q > 0$ fixed, the distribution of $\mathcal{T}_q$ under $\mathbb{N}^\psi$ is $\mathbb{N}^{\psi_q}$ with the branching mechanism ψ_q defined for $\lambda \geq 0$ by:*

$$\psi_q(\lambda) = \psi(\lambda + q) - \psi(q). \tag{14}$$

Furthermore, the measure $\mathbb{N}^{\psi_q}$ is absolutely continuous with respect to $\mathbb{N}^\psi$, see [2, Lemma 6.2]: for every $q \geq 0$ and every non-negative measurable function F on $\mathbb{T}$, we have

$$\mathbb{N}^{\psi_q}[F(\mathcal{T})] = \mathbb{N}^\psi\left[F(\mathcal{T})\mathrm{e}^{-\psi(q)\sigma}\right]. \tag{15}$$

We shall refer to this equation as the Girsanov transformation for Lévy trees as it corresponds to the Girsanov transformation of the height process (which is Brownian) in the quadratic case $\pi(dr) = 0$. This transformation corresponds also to the Esscher transformation for the underlying Lévy process used in [16] to define the height process in the general case. We deduce from definition (9) of $\mathbf{N}^\psi$, that for any measurable non-negative functionals F and $q \geq 0$:

$$\mathbf{N}^{\psi_q}[F(\mathcal{T})] = \mathbf{N}^\psi\left[F(\mathcal{T})\mathrm{e}^{-\psi(q)\sigma}\right]. \tag{16}$$

Making q vary allows us to define a tree-valued process $(\mathcal{T}_q, q \geq 0)$ which is a Markov process under $\mathbb{N}^\psi$; see [2, Lemma 5.3] stated for the family of exploration processes which codes for the corresponding Lévy trees. The process $(\mathcal{T}_q, q \geq 0)$ is a non-increasing process (for the inclusion of trees), and is càdlàg. Its one-dimensional marginals are described in Proposition 2.2 whereas its transition probabilities are given by the so-called special Markov property (see [6, Theorem 4.2] or [2, Theorem 5.6]). The time-reversed process is also a Markov process and its infinitesimal transitions are described in [4] using a point process whose definition we recall now. We set:

$$\{\theta_i, i \in I^R\}$$

the set of jumping times of the process $(\mathcal{T}_\theta, \theta \geq 0)$. For every $i \in I^R$, we set $\mathcal{T}^{i,\circ} = \mathcal{T}_{\theta_i-} \setminus \mathcal{T}_{\theta_i}$ and denote by x_i the MRCA of $\mathcal{T}^{i,\circ}$. For $i \in I^R$, we set:

$$\mathcal{T}^i = \mathcal{T}^{i,\circ} \cup \{x_i\}$$

which is a real tree with distance the induced distance, root x_i and mass measure the restriction of $\mathbf{m}^{\mathcal{T}}$ to $\mathcal{T}^i$. Finally, we define, conditionally on $\mathcal{T}_0$, the following random point measure on $\mathcal{T}_0 \times \mathbb{T} \times \mathbb{R}_+$:

$$\mathcal{N} = \sum_{i \in I^R} \delta_{(x_i, \mathcal{T}^i, \theta_i)}.$$

Theorem 2.3 ([4, *Theorem 3.2 and Lemma 3.3*]). *Under $\mathbb{N}^\psi$, the predictable compensator of the backward point process defined on $\mathbb{R}_+$ by:*

$$\theta \mapsto \mathbf{1}_{\{\theta \leq q'\}} \mathcal{N}(dx, d\mathcal{T}, dq')$$

with respect to the backward left-continuous filtration $\mathcal{F} = (\mathcal{F}_\theta, \theta \geq 0)$ defined by:

$$\mathcal{F}_\theta = \sigma((x_i, \mathcal{T}^i, \theta_i), i \in I^R, \theta_i \geq \theta) = \sigma(\mathcal{T}_{q-}, q \geq \theta)$$

is given by:

$$\mu(dx, d\mathcal{T}, dq) = \mathbf{m}^{\mathcal{T}_q}(dx) \mathbf{N}^{\psi_q}[d\mathcal{T}] \mathbf{1}_{\{q \geq 0\}} dq.$$

And for any non-negative predictable process ϕ with respect to the backward filtration $\mathcal{F}$, we have:

$$\mathbb{N}^\psi \left[\int \mathcal{N}(dx, d\mathcal{T}, dq) \phi(q, \mathcal{T}_q, \mathcal{T}_{q-}) \right] = \mathbb{N}^\psi \left[\int \mu(dx, dT, dq) \phi(q, \mathcal{T}_q, \mathcal{T}_q \circledast (T, x)) \right].$$

3. Statement of the main result

We keep the notations of the previous section. First notice that for $i \in I^R$, $\theta(x) = \theta_i$ for every $x \in \mathcal{T}^i$. We set $\sigma^i = \mathbf{m}^{\mathcal{T}}(\mathcal{T}^i) = \sigma_{\theta_i-} - \sigma_{\theta_i}$ and $\sigma_q = \mathbf{m}^{\mathcal{T}}(\mathcal{T}_q)$ the total mass of $\mathcal{T}_q$. By construction, we have for every $q \geq 0$:

$$\sigma_q = \sum_{i \in I^R} \mathbf{1}_{\{\theta_i \geq q\}} \sigma^i.$$

We set:

$$\Theta_q = \int_{\mathcal{T}_q} (\theta(x) - q)\, \mathbf{m}^{\mathcal{T}}(dx).$$

The quantity $\Theta := \Theta_0$ appears in [1] as the limit of the number of cuts on Aldous's CRT to isolate the root. Since $\theta(x)$ is constant on $\mathcal{T}^i$, we get:

$$\Theta_q = \sum_{i \in I^R} \mathbf{1}_{\{\theta_i \geq q\}} (\theta_i - q) \sigma^i = \int_q^{+\infty} \sigma_r \, dr.$$

For simplicity, we write Θ for Θ_0 and σ for σ_0.

We consider the random point measure $\mathcal{Z}^R$ on $\mathbb{R}_+ \times \mathbb{T}$ defined by:

$$\mathcal{Z}^R = \sum_{i \in I^R} \delta_{(\Theta_{\theta_i}, \mathcal{T}^i)}. \tag{17}$$

Recall the definition of H and $\mathcal{Z}^B$ of Section 2.7.

The main result of the paper is the next theorem that identifies the law of the pair $(H, \mathcal{Z}^B)$ and the pair $(\Theta, \mathcal{Z})$.

Theorem 3.1. *Assume that the Grey condition holds. For every non-negative measurable function Φ on $\mathbb{R}_+ \times \mathbb{T}$, and every $\lambda > 0$, $\rho \geq 0$, we have:*

$$\mathbb{N}^\psi \left[\sigma \mathrm{e}^{-\lambda\sigma - \rho H - \langle \mathcal{Z}^B, \Phi \rangle} \right] = \mathbb{N}^\psi \left[\sigma \mathrm{e}^{-\lambda\sigma - \rho \Theta - \langle \mathcal{Z}^R, \Phi \rangle} \right].$$

In particular Θ is distributed as the height H of a leaf chosen according to the normalized mass measure on the Lévy tree.

Recall that $\lim_{\varepsilon \to 0} \mathbb{N}^\psi[\sigma > \varepsilon] = +\infty$ and $\lim_{\varepsilon \to 0} \mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma \leq \varepsilon\}}] = 0$, as well as:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma \leq \varepsilon\}}] = +\infty$$

thanks to Lemma 4.1 from [13] (which is stated for $\beta = 0$ but which also holds for $\beta > 0$). The next corollary is a direct consequence of Theorem 3.1 and the properties of Poisson point measures for the Bismut decomposition (see Proposition 4.2 in [13] for a proof of similar results).

Corollary 3.2. *Assume that the Grey condition holds. $\mathbb{N}^\psi$-a.e., we have:*

$$\lim_{\varepsilon \to 0} \frac{1}{\mathbb{N}^\psi[\sigma > \varepsilon]} \sum_{i \in I^R} \mathbf{1}_{\{\sigma^i \geq \varepsilon\}} = \Theta.$$

$\mathbb{N}^\psi$-a.e., for any positive sequence $(\varepsilon_n, n \geq 0)$ converging to 0, there exists a subsequence $(\varepsilon_{n_k}, k \geq 0)$ such that:

$$\lim_{k \to +\infty} \frac{1}{\mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma \leq \varepsilon_{n_k}\}}]} \sum_{i \in I^R} \sigma^i \mathbf{1}_{\{\sigma^i \leq \varepsilon_{n_k}\}} = \Theta.$$

When ψ is regularly varying at infinity with index $\gamma \in (1, 2]$, the previous convergence holds $\mathbb{N}^\psi$-a.e.

4. Proof of the main result

4.1. Preliminary results

We first state a basic lemma.

Lemma 4.1. *Let $\mathcal{N}_1 = \sum_{j \in J_1} \delta_{r_j, x_j}$ be a point measure on $[0, +\infty)$. If $\sum_{j \in J_1} x_j < +\infty$, then for every $r \geq 0$, we have:*

$$1 - \exp\left(- \sum_{j \in J_1} \mathbf{1}_{\{r_j \geq r\}} x_j \right) = \sum_{j \in J_1} \mathbf{1}_{\{r_j \geq r\}} (1 - \mathrm{e}^{-x_j}) \exp\left(- \sum_{\ell \in J_1} \mathbf{1}_{\{r_\ell > r_j\}} x_\ell \right). \tag{18}$$

Proof. The result is obvious for J_1 finite. For the infinite case, for $\varepsilon > 0$ consider the finite set:

$$J_{1,\varepsilon} = \{j \in J_1,\ x_j \geq \varepsilon\}.$$

Apply Formula (18) with J_1 replaced by $J_{1,\varepsilon}$ and then conclude by letting ε tend to 0 thanks to monotone convergence and dominated convergence. $\square$

Since $\mathcal{T}_q$ is distributed according to $\mathbb{N}^{\psi_q}$, we deduce from (8) together with (16) that for $q > 0$:

$$\mathbb{N}^{\psi}[\sigma_q] = \mathbb{N}^{\psi_q}[\sigma] = \frac{1}{\psi'(q)}, \qquad \mathbb{N}^{\psi}[\sigma_q^2] = \mathbb{N}^{\psi_q}[\sigma^2] = \frac{\psi''(q)}{\psi'(q)^3}. \tag{19}$$

4.2. Laplace transform of $(\sigma, \Theta, \mathcal{Z}^R)$

Proposition 4.2. *Let Φ be a non-negative measurable function on $\mathbb{R}_+ \times \mathbb{T}$. Assume that $\langle \mathcal{Z}^R, \Phi\rangle < +\infty$ $\mathbb{N}^{\psi}$-a.e. and for all $\lambda > 0$, $\sup_{u\geq 0} g(\lambda, u) < +\infty$ with g defined by (13). Then, for all $\lambda > 0$ and $\rho \geq 0$, we have:*

$$\mathbb{N}^{\psi}\left[\sigma\,(\rho + g(\lambda, \Theta))\,\mathrm{e}^{-\lambda\sigma - \rho\Theta - \langle \mathcal{Z}^R, \Phi\rangle}\right] = 1. \tag{20}$$

Proof. For every $\varepsilon > 0$, $q \geq 0$, we set:

$$\sigma_q^{\varepsilon} = \sum_{i\in I^R} \mathbf{1}_{\{\theta_i \geq q\}} \mathbf{1}_{\{\sigma^i \geq \varepsilon\}} \sigma^i, \qquad \Theta_q^{\varepsilon} = \sum_{i\in I^R} \mathbf{1}_{\{\theta_i \geq q\}} \mathbf{1}_{\{\sigma^i \geq \varepsilon\}} \sigma^i (\theta_i - q),$$

and

$$Z_q^{\varepsilon} = \sum_{i\in I^R} \mathbf{1}_{\{\theta_i \geq q\}} \mathbf{1}_{\{\sigma^i \geq \varepsilon\}} \Phi(\Theta_{\theta_i}, \mathcal{T}^i), \qquad Z_q = \sum_{i\in I^R} \mathbf{1}_{\{\theta_i \geq q\}} \Phi(\Theta_{\theta_i}, \mathcal{T}^i),$$

so that $Z_0 = \langle \mathcal{Z}^R, \Phi\rangle$. For every $\varepsilon > 0$, $q > 0$, we set:

$$\varphi_q^{\varepsilon}(\lambda, \rho) = \mathbb{N}^{\psi}\left[1 - \exp(-\lambda\sigma_q^{\varepsilon} - \rho\Theta_q^{\varepsilon} - Z_q^{\varepsilon})\right].$$

Since $\langle \mathcal{Z}^R, \Phi\rangle$ is finite by assumption, we get that Z_q^{ε} is finite. We use Lemma 4.1 to get:

$$\begin{aligned}\varphi_q^{\varepsilon}(\lambda, \rho) = \mathbb{N}^{\psi}\Bigg[&\sum_{i\in I^R} \mathbf{1}_{\{\theta_i \geq q\}} \mathbf{1}_{\{\sigma^i \geq \varepsilon\}} \left(1 - \exp\left(-\left(\lambda + \rho(\theta_i - q)\right)\sigma^i - \Phi(\Theta_{\theta_i}, \mathcal{T}^i)\right)\right) \\ &\times \exp\left(-\sum_{\ell\in I^R} \mathbf{1}_{\{\theta_\ell > \theta_i\}} \mathbf{1}_{\{\sigma^\ell \geq \varepsilon\}} \left(\left(\lambda + \rho(\theta_\ell - q)\right)\sigma^\ell + \Phi(\Theta_{\theta_\ell}, \mathcal{T}^\ell)\right)\right)\Bigg].\end{aligned}$$

Then, if we use Theorem 2.3 (recall that $\sigma_q = \mathbf{m}^{\mathcal{T}_q}(\mathcal{T}_q)$), we get:

$$\begin{aligned}\varphi_q^{\varepsilon}(\lambda, \rho) = \mathbb{N}^{\psi}\Bigg[&\int_q^{+\infty} dr\, \sigma_r\, G_r^{\varepsilon}(\lambda + \rho(r - q), \Theta_r) \\ &\times \exp\left(-\left(\lambda + \rho(r - q)\right)\sigma_r^{\varepsilon} - \rho\Theta_r^{\varepsilon} - Z_r^{\varepsilon}\right)\Bigg],\end{aligned}$$

with

$$G_r^{\varepsilon}(\kappa, t) = \mathbf{N}^{\psi_r}\left[\mathbf{1}_{\{\sigma \geq \varepsilon\}}\left(1 - \mathrm{e}^{-\kappa\sigma - \Phi(t, \mathcal{T})}\right)\right].$$

Thanks to (11) and (16), we get:

$$0 \le G_r^\varepsilon(\kappa, t) \le \mathbf{N}^{\psi_r}\left[\mathbf{1}_{\{\sigma \ge \varepsilon\}}\right] \le \frac{1}{\varepsilon}\mathbf{N}^{\psi_r}[\sigma] = \frac{1}{\varepsilon}\frac{\psi''(r)}{\psi'(r)}. \tag{21}$$

Since ψ'' is non-increasing and ψ' is non-decreasing, we get that for fixed $q > 0$, the map $r \mapsto \partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right)$ is non-negative and bounded for $r > q$. We deduce from (11) and (16) that:

$$\mathbf{N}^{\psi_r}\left[\mathbf{1}_{\{\sigma \ge \varepsilon\}}\sigma \mathrm{e}^{-\kappa\sigma - \Phi(t,\mathcal{T})}\right] \le \frac{1}{\varepsilon}\mathbf{N}^{\psi_r}\left[\sigma^2\right] = \frac{1}{\varepsilon}\frac{1}{\psi'(r)}\partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right).$$

We deduce that the map $\kappa \mapsto G_r^\varepsilon(\kappa, t)$ is $\mathcal{C}^1$ and:

$$0 \le \partial_\kappa G_r^\varepsilon(\kappa, t) = \mathbf{N}^{\psi_r}\left[\mathbf{1}_{\{\sigma \ge \varepsilon\}}\sigma \mathrm{e}^{-\kappa\sigma - \Phi(t,\mathcal{T})}\right] \le \frac{1}{\varepsilon}\frac{1}{\psi'(r)}\partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right). \tag{22}$$

We set:

$$H_{r,\lambda}^\varepsilon(q) = \mathbb{N}^\psi\left[\sigma_r\, G_r^\varepsilon(\lambda + \rho(r-q), \Theta_r)\, \exp\left(-(\lambda + \rho(r-q))\,\sigma_r^\varepsilon - \rho\,\Theta_r^\varepsilon - Z_r^\varepsilon\right)\right],$$

so that:

$$\varphi_q^\varepsilon(\lambda, \rho) = \int_q^{+\infty} H_{r,\lambda}^\varepsilon(q)\, dr.$$

Thanks to (21) and (19), we get $0 \le H_{r,\lambda}^\varepsilon(q) \le \varepsilon^{-1}\psi''(r)/\psi'(r)^2$. This implies in turn that $\varphi_q^\varepsilon(\lambda, \rho) \le \varepsilon^{-1}/\psi'(q)$.

For $r > 0$, $\kappa > 0$, we set:

$$h_r^\varepsilon(\kappa) = \mathbb{N}^\psi\left[\sigma_r\,\left(\partial_\kappa G_r^\varepsilon(\kappa, \Theta_r) + \sigma_r^\varepsilon G_r^\varepsilon(\kappa, \Theta_r)\right)\, \mathrm{e}^{-\kappa\sigma_r^\varepsilon - \rho\Theta_r^\varepsilon - Z_r^\varepsilon}\right].$$

Since $\sigma_r^\varepsilon \le \sigma_r$, we have, using (19):

$$\begin{aligned} 0 \le h_r^\varepsilon(\kappa) &\le \frac{1}{\varepsilon}\mathbb{N}^\psi\left[\sigma_r\frac{1}{\psi'(r)}\partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right) + \sigma_r^2\frac{\psi''(r)}{\psi'(r)}\right] \\ &\le \frac{1}{\varepsilon}\left[\frac{1}{\psi'(r)^2}\partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right) + \frac{\psi''(r)^2}{\psi'(r)^4}\right]. \end{aligned}$$

By monotonicity, we get:

$$\begin{aligned} &\int_{[q,+\infty)^2} du ds\, \mathbf{1}_{\{u<s\}} h_s^\varepsilon(\lambda + \rho(s-u)) \\ &\quad\le \int_{[q,+\infty)^2} du ds\, \mathbf{1}_{\{u<s\}}\frac{1}{\varepsilon}\left[\frac{1}{\psi'(s)^2}\partial_s\left(\frac{-\psi''(s)}{\psi'(s)}\right) + \frac{\psi''(s)^2}{\psi'(s)^4}\right] \\ &\quad\le \int_{[q,+\infty)^2} du ds\, \frac{1}{\varepsilon}\mathbf{1}_{\{u<s\}}\left[\frac{1}{\psi'(u)^2}\partial_s\left(\frac{-\psi''(s)}{\psi'(s)}\right) + \frac{\psi''(u)}{\psi'(u)^2}\frac{\psi''(s)}{\psi'(s)^2}\right] \\ &\quad= \frac{2}{\varepsilon}\int_{[q,+\infty)} du\, \frac{\psi''(u)}{\psi'(u)^3} \\ &\quad= \frac{1}{\varepsilon}\frac{1}{\psi'(q)^2}. \end{aligned}$$

We deduce that the maps $u \mapsto H^\varepsilon_{s,\lambda}(u)$ and $\lambda \mapsto H^\varepsilon_{s,\lambda}(u)$ are $\mathcal{C}^1$ for $\lambda \geq 0$, $s \geq u \geq q$, with:

$$\partial_u H^\varepsilon_{s,\lambda}(u) = -\rho \partial_\lambda H^\varepsilon_{s,\lambda}(u) \quad \text{and} \quad \left|\partial_\lambda H^\varepsilon_{s,\lambda}(u)\right| \leq h^\varepsilon_s(\lambda + \rho(s-u)).$$

Thus we have $\int_{[q,+\infty)^2} du ds\, \mathbf{1}_{\{u<s\}} \left|\partial_u H^\varepsilon_{s,\lambda}(u)\right| \leq \rho/\varepsilon\psi'(q)^2$. Then, elementary computation yields:

$$\varphi^\varepsilon_q(\lambda,\rho) = \int_q^{+\infty} H^\varepsilon_{r,\lambda}(q)\, dr = \int_q^{+\infty} du \left[H^\varepsilon_{u,\lambda}(u) - \int_u^{+\infty} ds\, \partial_u H^\varepsilon_{s,\lambda}(u)\right].$$

We deduce that the maps $q \mapsto \varphi^\varepsilon_q(\lambda,\rho)$ and $\lambda \mapsto \varphi^\varepsilon_q(\lambda,\rho)$ are $\mathcal{C}^1$ and:

$$\partial_q \varphi^\varepsilon_q(\lambda,\rho) = -H^\varepsilon_{q,\lambda}(q) + \int_q^{+\infty} ds\, \partial_u H^\varepsilon_{s,\lambda}(q) = -H^\varepsilon_{q,\lambda}(q) - \rho\partial_\lambda \int_q^{+\infty} ds\, H^\varepsilon_{s,\lambda}(q).$$

With $H^\varepsilon_{q,\lambda}(q) = \mathbb{N}^\psi\left[\sigma_q\, G^\varepsilon_q(\lambda,\Theta_q)\exp\left(-\lambda\sigma^\varepsilon_q - \rho\Theta^\varepsilon_q - Z^\varepsilon_q\right)\right]$, we deduce that:

$$\partial_q \varphi^\varepsilon_q(\lambda,\rho) = -\mathbb{N}^\psi\left[\sigma_q\, G^\varepsilon_q(\lambda,\Theta_q)\exp\left(-\lambda\sigma^\varepsilon_q - \rho\Theta^\varepsilon_q - Z^\varepsilon_q\right)\right] - \rho\partial_\lambda\varphi^\varepsilon_q(\lambda,\rho). \tag{23}$$

We also have:

$$\partial_\lambda \varphi^\varepsilon_q(\lambda,\rho) = \mathbb{N}^\psi\left[\sigma^\varepsilon_q \exp(-\lambda\sigma^\varepsilon_q - \rho\Theta^\varepsilon_q - Z^\varepsilon_q)\right]. \tag{24}$$

Moreover, thanks to Girsanov formula (15), we have:

$$\varphi^\varepsilon_q(\lambda,\rho) = \mathbb{N}^\psi\left[\left(1 - \exp(-\lambda\sigma^\varepsilon_0 - \rho\Theta^\varepsilon_0 - Z^\varepsilon_0)\right) e^{-\psi(q)\sigma}\right].$$

We deduce that:

$$\begin{aligned}\partial_q \varphi^\varepsilon_q(\lambda,\rho) &= -\psi'(q)\mathbb{N}^\psi\left[\sigma\left(1 - \exp(-\lambda\sigma^\varepsilon_0 - \rho\Theta^\varepsilon_0 - Z^\varepsilon_0)\right) e^{-\psi(q)\sigma}\right] \\ &= -1 + \psi'(q)\mathbb{N}^\psi\left[\sigma_q \exp(-\lambda\sigma^\varepsilon_q - \rho\Theta^\varepsilon_q - Z^\varepsilon_q)\right].\end{aligned}$$

We deduce from (23) and (24) that:

$$\mathbb{N}^\psi\left[\left(\sigma_q(\psi'(q) + G^\varepsilon_q(\lambda,\Theta_q)) + \rho\sigma^\varepsilon_q\right)\exp\left(-\lambda\sigma^\varepsilon_q - \rho\Theta^\varepsilon_q - Z^\varepsilon_q\right)\right] = 1. \tag{25}$$

Using Girsanov formulae (16) and (10), we get:

$$\begin{aligned}G^\varepsilon_q(\lambda,t) \leq G^0_q(\lambda,t) &= g(\lambda + \psi(q), t) - \psi'(0) - \mathbf{N}^\psi\left[1 - e^{-\psi(q)\sigma}\right] \\ &= g(\lambda + \psi(q), t) - \psi'(q).\end{aligned}$$

We deduce that:

$$\sigma_q(\psi'(q) + G^\varepsilon_q(\lambda,\Theta_q)) + \rho\sigma^\varepsilon_q \leq \sigma_q\left(\sup_{t\geq 0} g(\lambda + \psi(q), t) + \rho\right).$$

By dominated convergence, letting ε decrease to 0 in (25), we deduce that:

$$\mathbb{N}^\psi\left[\sigma_q\left(g(\lambda + \psi(q), \Theta) + \rho\right)\exp\left(-\lambda\sigma_q - \rho\Theta_q - Z_q\right)\right] = 1.$$

Using Girsanov formula (15) once again, we get:

$$\mathbb{N}^{\psi}\left[\sigma\,(g(\lambda+\psi(q),\,\Theta)+\rho)\,\exp\left(-(\lambda+\psi(q))\sigma-\rho\Theta-\langle\mathcal{Z}^R,\,\Phi\rangle\right)\right]=1.$$

Since $\lambda > 0$ and $q > 0$ are arbitrary, we deduce that (20) holds. □

We deduce the following corollary which states that the pair H and the projection of $\mathcal{Z}^B$ on $\mathbb{T}$ have the same distribution as Θ and the projection of $\mathcal{Z}^R$ on $\mathbb{T}$.

Let γ be a non-negative measurable function defined on $\mathbb{T}$. For a measure $\mathcal{Z}$ on $\mathbb{R}_+ \times \mathbb{T}$, we shall abuse notation and write:

$$\langle\mathcal{Z},\gamma\rangle=\int\gamma(\mathcal{T})\,\mathcal{Z}(dt,d\mathcal{T}).$$

Corollary 4.3. *For every non-negative measurable function γ on $\mathbb{T}$ such that $\gamma(\mathcal{T}) = 0$ if $\mathbf{m}^{\mathcal{T}}(\mathcal{T}) = 0$, and every $\lambda \geq 0$, $\rho \geq 0$, we have:*

$$\mathbb{N}^{\psi}\left[\sigma\mathrm{e}^{-\lambda\sigma-\rho H-\langle\mathcal{Z}^B,\gamma\rangle}\right]=\mathbb{N}^{\psi}\left[\sigma\mathrm{e}^{-\lambda\sigma-\rho\Theta-\langle\mathcal{Z}^R,\gamma\rangle}\right].\tag{26}$$

Proof. Let $\lambda > 0$. Recall $\sigma = \mathbf{m}^{\mathcal{T}}(\mathcal{T})$. First assume that $\gamma(\mathcal{T}) \leq c\sigma$ for some finite constant c. Taking $\Phi(t,\mathcal{T}) = \gamma(\mathcal{T})$ in Theorem 2.1 and using that $g(\lambda, u)$ does not depend on u, we get:

$$\mathbb{N}^{\psi}\left[\sigma\mathrm{e}^{-\lambda\sigma-\rho H-\langle\mathcal{Z}^B,\gamma\rangle}\right]=\frac{1}{\rho+g(\lambda,0)}.$$

Notice that $\langle\mathcal{Z}^R,\,\Phi\rangle \leq c\sigma$ and thus hypotheses from Proposition 4.2 are in force. We deduce from Proposition 4.2 that:

$$\mathbb{N}^{\psi}\left[\sigma\,\exp\left(-\lambda\sigma-\rho\Theta-\langle\mathcal{Z}^R,\,\gamma\rangle\right)\right]=\frac{1}{\rho+g(\lambda,0)}.$$

Thus equality (26) holds. Use monotone convergence to remove hypotheses $\lambda > 0$ and $\gamma(\mathcal{T}) \leq c\sigma$ for some finite constant c. □

4.3. Proof of Theorem 3.1

Let Φ be a measurable non-negative function defined on the space $\mathbb{R}_+ \times \mathbb{T}$. Let us assume that for every $\mathcal{T} \in \mathbb{T}$, $t \mapsto \Phi(t,\mathcal{T})$ is continuous, $\langle\mathcal{Z}^R,\,\Phi\rangle$ is finite $\mathbb{N}^{\psi}$-a.s. and that the function g defined by (13) is bounded for any $\lambda > 0$ as a function of u. We set:

$$\Gamma^R(r,h)=\mathbb{N}^{\psi}\left[\mathrm{e}^{-\langle\mathcal{Z}^R,\Phi\rangle}\middle|\,\sigma=r,\,\Theta=h\right].$$

We deduce from Proposition 4.2 and Corollary 4.3 that for every $\lambda > 0$, $\rho \geq 0$, we have:

$$\begin{aligned}1&=\mathbb{N}^{\psi}\left[\sigma\,(\rho+g(\lambda,\,\Theta))\,\mathrm{e}^{-\lambda\sigma-\rho\Theta-\langle\mathcal{Z}^R,\Phi\rangle}\right]\\&=\mathbb{N}^{\psi}\left[\sigma\,(\rho+g(\lambda,\,\Theta))\,\mathrm{e}^{-\lambda\sigma-\rho\Theta}\,\Gamma^R(\sigma,\,\Theta)\right]\\&=\mathbb{N}^{\psi}\left[\sigma\,(\rho+g(\lambda,\,H))\,\mathrm{e}^{-\lambda\sigma-\rho H}\,\Gamma^R(\sigma,\,H)\right].\end{aligned}\tag{27}$$

Let $\sum_{i\in I}\delta_{(h_i,\mathcal{T}_i)}$ be a Poisson measure with intensity $dh\,\mathbf{N}^{\psi}[d\mathcal{T}]$ under some probability measure P. For every $i\in I$, we set $\sigma_i=\mathbf{m}^{\mathcal{T}_i}(\mathcal{T}_i)$. Then for every $h>0$, we set:

$$\sigma(h)=\sum_{i\in I}\mathbf{1}_{\{h_i\leq h\}}\sigma_i.$$

Eq. (27) and Theorem 2.1 imply that:

$$\int_0^{+\infty}dh\,\mathrm{e}^{-(\rho+\psi'(0))h}\mathrm{e}^{-G(h)}(\rho+g(\lambda,h))=1,$$

with:

$$G(h)=-\log\left(E\left[\mathrm{e}^{-\lambda\sigma(h)}\Gamma^R(\sigma(h),h)\right]\right).$$

We deduce that:

$$\int_0^{+\infty}dh\,\mathrm{e}^{-\rho h}\left[1-\mathrm{e}^{-\psi'(0)h-G(h)}\right]=\int_0^{+\infty}\frac{1}{\rho}\mathrm{e}^{-\rho h}\,dA(h)=\int_0^{+\infty}dh\,\mathrm{e}^{-\rho h}A(h),$$

with:

$$A(h)=\int_0^h du\,\mathrm{e}^{-\psi'(0)u-G(u)}g(\lambda,u).$$

Since this holds for every $\rho\geq 0$, uniqueness of the Laplace transform implies that:

$$A(h)=1-\mathrm{e}^{-\psi'(0)h-G(h)}\quad\text{a.e.}\tag{28}$$

Since A is continuous, there exists a continuous function $\tilde{G}$ such that a.e. $\tilde{G}=G$. Since, $t\mapsto\Phi(t,\mathcal{T})$ is continuous, we get that, for every $\lambda\geq 0$, $u\mapsto g(\lambda,u)$ is continuous. Then A is of class $\mathcal{C}^1$ and so is $\tilde{G}$. Moreover, by differentiating (28), we get:

$$\psi'(0)+\tilde{G}'(h)=g(\lambda,h).$$

Since $A(0)=0$, we get $\tilde{G}(0)=0$, and thus $\psi'(0)h+\tilde{G}(h)=\int_0^h g(\lambda,u)du$. This implies that:

$$\int_0^h g(\lambda,u)du=G(h)+\psi'(0)h\quad\text{a.e.}\tag{29}$$

We have:

$$\begin{aligned}\mathbb{N}^{\psi}\left[\sigma\mathrm{e}^{-\lambda\sigma-\rho H-\langle\mathcal{Z}^B,\Phi\rangle}\right]&=\int_0^{+\infty}dh\,\mathrm{e}^{-\rho h-\int_0^h g(\lambda,u)du}\\&=\int_0^{+\infty}dh\,\mathrm{e}^{-(\rho+\psi'(0))h-G(h)}\\&=\int_0^{+\infty}dh\,\mathrm{e}^{-(\rho+\psi'(0))h}E\left[\mathrm{e}^{-\lambda\sigma(h)}\Gamma^R(\sigma(h),h)\right]\\&=\mathbb{N}^{\psi}\left[\sigma\mathrm{e}^{-\lambda\sigma-\rho H}\Gamma^R(\sigma,H)\right]\\&=\mathbb{N}^{\psi}\left[\sigma\mathrm{e}^{-\lambda\sigma-\rho\Theta}\Gamma^R(\sigma,\Theta)\right]\\&=\mathbb{N}^{\psi}\left[\sigma\mathrm{e}^{-\lambda\sigma-\rho\Theta-\langle\mathcal{Z}^R,\Phi\rangle}\right],\end{aligned}$$

where we used Theorem 2.1 for the first and fourth equalities, (29) for the second, the definition of G for the third, Corollary 4.3 (which states that (σ, H) and (σ, Θ) have the same distribution under $\mathbb{N}^\psi$) for the fifth, and the definition of Γ^R for the last.

As $\mathbb{N}^\psi\left[\sigma e^{-\lambda\sigma}\right]$ is finite, we can remove using dominated convergence the hypothesis $\langle \mathcal{Z}^R, \Phi\rangle$ finite. The function g defined by (13), with $\Phi(t, \mathcal{T})$ replaced by $\Phi(t, \mathcal{T})\mathbf{1}_{\{\sigma\leq 1/n\}}$, is bounded for any $\lambda > 0$ as a function of u. Thus, using again dominated convergence, we can remove the hypothesis on Φ such that function g defined by (13) is bounded for any $\lambda > 0$ as a function of u. Then use monotone class theorem to remove the continuity hypothesis on Φ and end the proof.

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