EXIT TIMES FOR AN INCREASING LÉVY TREE-VALUED PROCESS

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Abstract. We extend some results from [3] about increasing Lévy tree-valued processes. First, we give an explicit construction of the tree-valued process using a Poisson point process of trees and a grafting procedure. We then use the Poissonian structure of the jumps of the process to describe its behavior at the first time the tree grows higher than a given height. This generalizes a result from [3] in which only the first infinite jump was considered.

1. Introduction

Lévy trees arise as a natural generalization to the continuum trees defined by Aldous [6]. They are located at the intersection of several important fields: combinatorics of large discrete trees, Lévy processes and branching processes. Given a (sub)critical branching mechanism ψ, that is, a function of the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}})\Pi(dx) \tag{1}$$

with $\beta \geq 0$, Π a Lévy measure and $\psi'(0) \geq 0$, Le Gall and Le Jan [21] defined a *continuum* tree structure, which can be described by a tree $\mathcal{T}$, for the genealogy of a population whose size is given by a CSBP with branching mechanism ψ. We shall consider the distribution of this Lévy tree, $\mathbb{P}^{\psi}_r(d\mathcal{T})$, when the CSBP starts at mass $r > 0$, or its excursion measure $\mathbb{N}^{\psi}[d\mathcal{T}]$, when the CSBP is distributed under its canonical measure. The ψ-Lévy tree possesses several striking features as pointed out in the works of Duquesne and Le Gall [10, 11]. For instance, the branching nodes can only be of degree 3 (binary branching) if $\beta > 0$ or of infinite degree (when removing the branching point, the tree is separated in infinitely many connected components) if $\Pi \neq 0$. Furthermore, there exists a mass measure, $\mathbf{m}^{\mathcal{T}}$, on the leaves of $\mathcal{T}$, whose total mass corresponds to the total population size, $\sigma = \mathbf{m}^{\mathcal{T}}(\mathcal{T})$, of the CSBP. We shall also consider the extinction time of the CSBP which corresponds to the height $H_{max}(\mathcal{T})$ of the tree $\mathcal{T}$. The results can be easily extended to the super-critical case, see Section 2.14, using a Girsanov transformation given by Abraham and Delmas [3].

In [3], a decreasing continuum tree-valued process is defined using the so-called pruning procedure of Lévy trees introduced in Abraham, Delmas and Voisin [5]. By marking a ψ-Lévy tree with two different kinds of marks (the first lying on the skeleton of the tree, the other on the nodes of infinite degree), one can prune the tree by throwing away all the points having a mark on the ancestral branch connecting them to the root. The main result of [5] is that the remaining tree is still a Lévy tree, with branching mechanism related to ψ. The idea of [3] is to consider a particular pruning with an intensity depending on a parameter θ, so that the corresponding branching mechanism, ψ_θ, is ψ shifted by θ:

$$\psi_\theta(\lambda) = \psi(\theta + \lambda) - \psi(\theta).$$

Letting θ vary enables to define a decreasing tree-valued Markov process $(\mathcal{T}_\theta, \theta \in \Theta^{\psi})$, with $\Theta^{\psi} \subset \mathbb{R}$ the set of θ for which ψ_θ is well-defined, and such that $\mathcal{T}_\theta$ is distributed according to $\mathbb{N}^{\psi_\theta}$. If we write

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σ_θ for the total mass of $\mathcal{T}_\theta$, then the process $(\sigma_\theta, \theta \in \Theta^\psi)$ is a pure-jump process. The case $\Pi = 0$ was studied by Aldous and Pitman [7]. The time-reversed tree-valued process is also a Markov process which defines a growing tree process. Let us mention that the same kind of ideas have been used by Aldous and Pitman [7] and by Abraham, Delmas and He [4] in the framework of Galton-Watson trees to define growing discrete tree-valued Markov processes.

In the discrete framework of [4], it is possible to define the infinitesimal transition rates of the growing tree process. In [16], Evans and Winter define another continuum tree-valued process using a prune and re-graft procedure. This process is reversible with respect to the law of Aldous's continuum random tree and its infinitesimal transitions are described using the theory of Dirichlet forms.

In this paper, we describe the infinitesimal behavior of the growing continuum tree-valued process, that is of $(\mathcal{T}_\theta, \theta \in \Theta^\psi)$ seen backwards in time. The Special Markov Property in [5] describes only two-dimensional distributions and hence the transition probabilities but, since the space of real trees is not locally compact, we cannot use the theory of infinitesimal generators to describe its infinitesimal transitions. Dirichlet forms cannot be used either since the process is not symmetric (it is increasing). However, it is a pure-jump process and our main result shows that the infinitesimal transitions of the process can be described using a random point process of trees which are grafted one by one on the leaves of the growing tree. More precisely, let $\{\theta_j; j \in J\}$ be the set of jumping times of the mass process $(\sigma_\theta, \theta \in \Theta^\psi)$. Then, informally, at time θ_j, a tree $\mathcal{T}^j$ distributed according to $\mathbf{N}^{\psi_{\theta_j}}[\mathcal{T} \in \bullet]$, with:

$$\mathbf{N}^{\psi_\theta}[\mathcal{T} \in \bullet] = 2\beta \mathbb{N}^{\psi_\theta}[\mathcal{T} \in \bullet] + \int_{(0,+\infty)} \Pi(dr) r\, \mathrm{e}^{-\theta r}\, \mathbb{P}_r^{\psi_\theta}(\mathcal{T} \in \bullet),$$

is grafted at x_j, a leaf of $\mathcal{T}_{\theta_j}$ chosen at random (according to the mass measure $\mathbf{m}^{\mathcal{T}_{\theta_j}}$). We also prove that the random point measure $\mathcal{N} = \sum_{j\in J} \delta_{(x_j, \mathcal{T}^j, \theta_j)}$ has predictable compensator:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx) \mathbf{N}^{\psi_\theta}[d\mathcal{T}]\; \mathbf{1}_{\{\theta \in \Theta^\psi\}}\, d\theta$$

with respect to the backwards in time natural filtration of the process. See Corollary 3.4 for a precise statement. In particular, this representation allows to describe the ascension time or explosion time:

$$A = \inf\{\theta \in \Theta^\psi,\ \sigma_\theta < \infty\}$$

as $\inf\{\theta_j, \mathbf{m}^{\mathcal{T}^j}(\mathcal{T}^j) < \infty\}$, the first time (backward!) a tree with infinite mass is grafted. This representation is also used in Abraham and Delmas [1] on the asymptotics of the records on discrete random trees.

This structure, somewhat similar to the Poissonian structure of the jumps of a Lévy process (although in our case the structure is neither homogeneous nor independent), enables us to study the first passage of the growing tree-valued process above a given height:

$$\Theta_h = \sup\{\theta \in \Theta^\psi,\ H_{max}(\mathcal{T}_\theta) > h\}.$$

Under the Grey condition, $\int^{+\infty} \frac{du}{\psi(u)} < \infty$, (positive probability for the strong extinction of the CSBP), we give the joint distribution of (A, Θ_h), see Proposition 4.3. In particular, Θ_h goes to A as h goes to infinity: for h very large, with high probability the process up to A will not have crossed height h, so that the first jump to cross height h will correspond to the grafting time of the first infinite tree, which happens at the ascension time A.

We also give in Theorem 4.6 the joint distribution of $(\mathcal{T}_{\Theta_h -}, \mathcal{T}_{\Theta_h})$ the tree just after and just before the jumping time Θ_h. And we give a decomposition of $\mathcal{T}_{\Theta_h}$ along the ancestral branch of the leaf on which the overshooting tree is grafted, which is similar to the classical Bismut decomposition of Lévy trees. Conditionally on this distribution, the overshooting tree is then distributed as a regular Lévy tree, conditioned on being high enough to perform the overshooting. This generalizes results in [5] about the ascension time of the tree-valued process. Notice that this approach could be easily generalized to study exit times of growing families of super-Brownian motions.

All the results of this paper are stated in terms of real trees and not in terms of the height process or the exploration process that encode the tree as in [5]. For this purpose, we define in Section 2.2 the state space of rooted real trees with a mass measure (called here weighted trees or w-trees) endowed with the so-called Gromov-Hausdorff-Prokhorov metric which is a slight generalization of the Gromov-Prokhorov metric introduced by Greven, Pfaffelhuber and Winter on the space of metric spaces with a probability mass measure. For our Lévy tree process to live in that space, we must consider branching mechanisms that satisfy the Grey condition which implies in the (sub)critical case that the corresponding height process is continuous and that the tree is compact. However, the tree-valued process is defined in [5] without this assumption and we conjecture that the jump representation of the tree-valued Markov process holds without this assumption.

The paper is organized as follows. In Section 2, we introduce all the material for our study: the state space of weighted real trees and the metric on it, see Section 2.2 ; the definition of sub(critical) Lévy trees via the height process ; the extension of the definition to super-critical Lévy trees ; the pruning procedure of Lévy trees. In Section 3, we recall the definition of the growing tree-valued process by the pruning procedure as in [5] in the setting of real trees and give another construction using the grafting of trees given by Poisson point processes. We prove in Theorem 3.2 that the two definitions agree. Section 4 is devoted to the application of this construction on the distribution of the tree at the times it overshoots a given height and just before, see Theorem 4.6.

2. The pruning of Lévy trees

2.1. **Real trees.** The first definitions of continuum random trees go back to Aldous [6]. Later, Evans, Pitman and Winter [15] used the framework of real trees, previously used in the context of geometric group theory, to describe continuum trees. We refer to [14, 20] for a general presentation of random real trees. Informally, real trees are metric spaces without loops, locally isometric to the real line.

More precisely, a metric space (T, d) is a *real tree* (or $\mathbb{R}$-*tree*) if the following properties are satisfied:

(1) For every $s, t \in T$, there is a unique isometric map $f_{s,t}$ from $[0, d(s, t)]$ to T such that $f_{s,t}(0) = s$ and $f_{s,t}(d(s, t)) = t$.
(2) For every $s, t \in T$, if q is a continuous injective map from $[0, 1]$ to T such that $q(0) = s$ and $q(1) = t$, then $q([0, 1]) = f_{s,t}([0, d(s, t)])$.

We say that a real tree is *rooted* if there is a distinguished vertex $\emptyset$, which will be called the *root* of T. Such a real tree is noted $(T, d, \emptyset)$.

If $s, t \in T$, we will note $[\![s, t]\!]$ the range of the isometric map $f_{s,t}$ described above. We will also note $[\![s, t[\![$ for the set $[\![s, t]\!] \setminus \{t\}$.

We give some vocabulary on real trees, which will be used constantly when dealing with Lévy trees. Let T be a real tree. If $x \in T$, we shall call *degree* of x, and note by $n(x)$, the number of connected components of the set $T \setminus \{x\}$. In a general tree, this number can be infinite, and this will actually be the case with Lévy trees. The set of *leaves* is defined as:

$$\mathrm{Lf}(T) = \{x \in T\backslash\{\emptyset\},\ n(x) = 1\}.$$

If $n(x) \geq 3$, we say that x is a *branching point*. The set of branching points will be noted $\mathrm{Br}(T)$. Among those, there is the set of *infinite* branching points, defined by

$$\mathrm{Br}_\infty(T) = \{x \in \mathrm{Br}(T),\ n(x) = \infty\}.$$

Finally, the *skeleton* of a real tree, noted $\mathrm{Sk}(T)$, is the set of points in the tree that aren't leaves. It should be noted, following Evans, Pitman and Winter [15], that the trace of the Borel σ-field of T on

$\mathrm{Sk}(T)$ is generated by the sets $[\![s,s']\!]$, $s,s'\in\mathrm{Sk}(T)$. Hence, it is possible to define a σ-finite Borel measure l^T on T, such that

$$l^T(\mathrm{Lf}(T))=0 \quad\text{and}\quad l^T([\![s,s']\!])=d(s,s').$$

This measure will be called *length measure* on T. If x,y are two points in a rooted real tree $(T,d,\emptyset)$, then there is a unique point $z\in T$, called the Most Recent Common Ancestor (MRCA) of x and y such that $[\![\emptyset,x]\!]\cap[\![\emptyset,y]\!]=[\![\emptyset,z]\!]$. This vocabulary is an illustration of the genealogical vision of real trees, in which the root is seen as the ancestor of the population represented by the tree. Similarly, if $x\in T$, we shall call *height* of x, and note by H_x the distance $d(\emptyset,x)$ to the root. The function $x\mapsto H_x$ is continuous on T, and we define the height of T:

$$H_{max}(T)=\sup_{x\in T}H_x.$$

2.2. **Gromov-Prokhorov metric.** The methods of this section are inspired by [12], but for the fact that we include measures on the trees, in the spirit of [23]. The spaces we consider will always be *rooted*, which means we specify one point of the space, called the root.

Let (X,d) be a Polish metric space. We will use the notation $\mathcal{M}_f(X)$ for the space of all finite Borel measures on X. If $A,B\in\mathcal{B}(X)$ are compact, we set

$$d_H^X(A,B)=\inf\{\varepsilon>0,\ A\subset B^\varepsilon \text{ and } B\subset A^\varepsilon\},$$

the Hausdorff distance between A and B, where $A^\varepsilon=\{x\in X,\inf_{y\in A}d(x,y)<\varepsilon\}$ is the $\varepsilon-$ *halo set* of A. It is classical that the space of compact subsets of X, endowed with the Hausdorff distance, is compact. Similarly, if $\mu,\nu\in\mathcal{M}_f(X)$, we set

$$d_P^X(\mu,\nu)=\inf\{\varepsilon>0,\ \mu(A)\le\nu(A^\varepsilon)+\varepsilon \text{ and } \nu(A)\le\mu(A^\varepsilon)+\varepsilon \text{ for all closed } A\},$$

the Prokhorov distance between μ and ν. It is also classical that the space of finite measures on X, endowed with the Prokhorov distance, is a Polish metric space, and that the topology generated by d_P is exactly the topology of weak convergence (convergence against continuous bounded functionals). If $\Phi: X\to X'$ is a Borel map between two Polish metric spaces and if μ is a Borel measure on X, we will note $\Phi_*\mu$ the image measure on X' defined by $\Phi_*\mu(A)=\mu(\Phi^{-1}(A))$.

We will now use these two distances in order to compare any two compact metric spaces, endowed with a finite Borel measure, even if they are not subspaces of the same Polish metric space. To do this, we will use a procedure due to Gromov ([17]).

Definition 2.1. *Let $\mathcal{X}=(X,d,\emptyset,\mu)$ and $\mathcal{X}'=(X',d',\emptyset',\mu')$ be two compact, rooted metric spaces, endowed with finite Borel measures. The Gromov-Hausdorff-Prokhorov distance between them is defined by*

$$d_f(\mathcal{X},\mathcal{X}')=\inf_{\Phi,\Phi',Z}\left(d_H^Z(\Phi(X),\Phi'(X'))+d_Z(\Phi(\emptyset),\Phi'(\emptyset'))+d_P^Z(\Phi_*\mu,\Phi'_*\mu')\right), \tag{2}$$

where the infimum is taken over all isometric embeddings $\Phi: X\hookrightarrow Z$ and $\Phi': X'\hookrightarrow Z$ into some common Polish metric space (Z,d_Z).

Note that equation (2) does not actually define a metric, since two isometric metric spaces endowed with the same measure will be at distance 0. In order to avoid this difficulty, we will work with isometry classes of compact metric spaces, in the following sense:

Definition 2.2. *Let $\mathcal{X}=(X,d,\emptyset,\mu)$ and $\mathcal{X}'=(X',d',\emptyset',\mu')$ be two rooted metric spaces, endowed with Borel measures. We say that $\mathcal{X}$ and $\mathcal{X}'$ are* GHP-isometric *if there exists an isometry $\Phi: X\to X'$ such that $\Phi(\emptyset)=\emptyset'$ and such that $\Phi_*\mu=\mu'$. The space of GHP-isometry classes of compact metric spaces endowed with finite Borel measures will be noted $\mathbb{K}$.*

It is clear that the distance d_f between two points in $\mathbb{K}$ is well-defined. It can be checked that d_f does indeed satisfy all the axioms of a metric (the tricky part is to check that if $d_f(\mathcal{X},\mathcal{X}')=0$ then $\mathcal{X}=\mathcal{X}'$, as is done in [9] for the Gromov-Hausdorff metric and in [23] in the case of probability measures on compact metric spaces). However, this definition is not yet general enough, as we want to deal with trees that might not be compact, and the measures defined on them might not be finite.

Definition 2.3. *Let (X,d) be a metric space. We say that X is a* length space *if for every $x,y\in X$, we have*

$$d(x,y)=\inf L(\gamma),$$

where the infimum is taken over all rectifiable curves $\gamma:[0,1]\to X$ such that $\gamma(0)=x$ and $\gamma(1)=y$, and where $L(\gamma)$ is the length of the rectifiable curve γ.

Note in particular that real trees are always length spaces. The following theorem will be of particular importance, since it will enable us to localize our definitions, in order to define a distance between locally compact spaces using the distance d_f between compact spaces.

Theorem 2.4. *(Hopf-Rinow) Let X be a length space. If X is complete and locally compact, then every closed bounded subset of X is compact.*

Thus, for now we will assume completeness as well as local compactness for our length spaces.

Definition 2.5. *Let $\mathcal{X}=(X,d,\emptyset,\mu)$ and $\mathcal{X}'=(X',d',\emptyset',\mu')$ be two rooted locally compact, complete length spaces, along with locally finite measures (measures that are finite on every bounded Borel subset). We define the Gromov-Hausdorff distance between $\mathcal{X}$ and $\mathcal{X}'$ by:*

$$d_{GHP}(\mathcal{X},\mathcal{X}')=\int_0^\infty \mathrm{e}^{-r}\left(1\wedge d_f(\mathcal{X}^{(r)},\mathcal{X}'^{(r)})\right)\,dr, \tag{3}$$

where the space $\mathcal{X}^{(r)}$ is the compact metric space $B_X(\emptyset,r)$, rooted at $\emptyset$, along with the induced metric $d^{(r)}$ and the restriction $\mu^{(r)}$ of μ to $B_X(\emptyset,r)$.

It can be checked that, in the particular context of length spaces, the map $r\mapsto d_f(\mathcal{X}^{(r)},\mathcal{X}'^{(r)})$ is càdlàg and thus measurable, so that the integral is always well-defined. Furthermore, it is clear that the Gromov-Hausdorff-Prokhorov metric is well-defined on GHP-isometry classes of rooted locally compact, complete spaces endowed with a locally finite Borel measure.

We will now check that the function defined by (3) is indeed a metric on $\mathbb{L}$, the space of (GHP-isometry classes of) locally compact, complete length spaces with locally finite Borel measures. Sometimes, when the context is clear, we will abuse notations and write X for the equivalence class of $(X,d,\emptyset,\mu)$ and we will use the notations $d^X,\emptyset^X,\mu^X$ to denote the metric, root and measure on X.

Proposition 2.6. *The function d_{GHP} is a metric on $\mathbb{L}$.*

Although the metrics d_f and d_{GHP} are strongly related, it is not always the case for compact spaces that convergence in the d_f metric implies convergence in the d_{GHP} metric. This is actually a striking feature of length spaces:

Lemma 2.7. *Let $X,(X_n,\ n\geq 1)$ be elements of $\mathbb{K}\cap\mathbb{L}$ such that $d_f(X_n,X)\to_{n\to\infty}0$. Then, we also have $d_{GHP}(X_n,X)\to_{n\to\infty}0$.*

The following pre-compactness criterion is an generalization of the well-known compactness theorem for compact metric spaces (see for instance Theorem 7.4.15 in [9])

Proposition 2.8. *Let $\mathcal{C}$ be a subset of $\mathbb{L}$, such that for every $r\geq 0$, $\varepsilon>0$, there exists $C(r)>0$ and an integer $N(r,\varepsilon)\geq 1$, such that*

(1) *For any $X\in\mathcal{C}$, the closed ball $B_X(\emptyset,r)$ can be covered by at most $N(r,\varepsilon)$ balls of diameter ε.*

(2) *For any $X \in \mathcal{C}$, we have $\mu(B_X(\emptyset, r)) < C(r)$.*

Then, every sequence in $\mathcal{C}$ admits a subsequence that converges in the d_{GHP} topology.

This criterion is useful when proving the following theorem, which ensures that the space $\mathbb{L}$ is indeed a Polish space, as is expected from a space on which we want to define probability measures.

Theorem 2.9. *The metric space $(\mathbb{L}, d_{GHP})$ is separable and complete.*

Proof. In order to prove separability, we can notice that the set $\mathbb{K} \cap \mathbb{L}$ of compact length spaces endowed with finite measures is dense in $\mathbb{L}$, since we have for all $r > 0$, $d_{GHP}(X^{(r)}, X) \leq \mathrm{e}^{-r}$. Then, every $X \in \mathbb{K}$ can be approximated in the d_f topology by a sequence of metric spaces with finite cardinal, rational edge-lengths and rational weights. Hence, $(\mathbb{K} \cap \mathbb{L}, d_f)$ is separable, being a subspace of a separable metric space. According to Lemma 2.7, $(\mathbb{K} \cap \mathbb{L}, d_{GHP})$ is also separable, which ends the proof.

As far as completeness is concerned, let $(X_n,\ n \geq 1)$ be a Cauchy sequence with respect to d_{GHP}. In order to prove that it converges, it is sufficient to prove that it is pre-compact. We will use Proposition 2.8. Let $r \geq 0$ and $\varepsilon > 0$. There exists $n_0 \geq 1$ such that for all $n, m \geq n_0$, we have $d_{GHP}(X_n, X_m) < \mathrm{e}^{-r}\,\varepsilon$. Thus, by definition, we have

$$\forall n, m\ \geq n_0,\ d_f(X_n^{(r)}, X_m^{(r)}) < \varepsilon.$$

This implies in particular that we have $d_{GH}(X_n^{(r)}, X_m^{(r)}) < \varepsilon$ and $\inf_{\Phi,\Phi',Z} d_P^Z(\Phi_*\mu_n^{(r)}, \Phi'_*\mu_m^{(r)}) < \varepsilon$. Following [12], from the first inequality, we get part (1) of Proposition 2.8.

From the second inequality, using the fact that the Prokhorov distance is always bounded below by the difference of total mass, and that total mass is invariant under isometries, we get that

$$\|\mu_n^{(r)}\| \leq \|\mu_{n_0}^{(r)}\| + \varepsilon,$$

which gives part (2) of Proposition 2.8. Thus the sequence (X_n) is precompact, so it converges, which ends the proof. □

We will use the definitions above in the particular context of complete locally compact rooted real trees. We already pointed out that such trees are always complete length spaces so the theory above applies.

Definition 2.10. *The space of (GHP-isometry classes) of complete locally compact rooted real trees endowed with locally finite Borel measures (in short,* w-trees*) will be noted $\mathbb{T}$.*

There is a nice characterization of real trees among complete connected spaces: they are the only ones that satisfy the so-called *four-point condition*:

$$\forall x_1, x_2, x_3, x_4 \in T,\ d(x_1, x_2) + d(x_3, x_4) \leq (d(x_1, x_3) + d(x_2, x_4)) \vee (d(x_1, x_4) + d(x_2, x_3)). \tag{4}$$

This implies that $\mathbb{T}$, as a subset of $\mathbb{L}$, is closed. Consequently, $(\mathbb{T}, d_{GHP})$ is a complete separable metric space.

2.3. **Height erasing.** We define the restriction operators on the space of w-trees. Let $a \geq 0$. If $(T, d, \emptyset, \mathbf{m})$ is a w-tree, let

$$\pi_a(T) = \{x \in T,\ d(\emptyset, x) \leq a\} \tag{5}$$

and $(\pi_a(T), d^{\pi_a(T)}, \emptyset, \mathbf{m}^{\pi_a(T)})$ be the w-tree constituted of the points of T having height lower than a, where $d^{\pi_a(T)}$ and $\mathbf{m}^{\pi_a(T)}$ are the restrictions of d and $\mathbf{m}$ to $\pi_a(T)$. When there is no confusion, we will also write $\pi_a(T)$ for $(\pi_a(T), d^{\pi_a(T)}, \emptyset, \mathbf{m}^{\pi_a(T)})$. We will also write $T(a) = \{x \in T,\ d(\emptyset, x) = a\}$ for the level set at height a. We say a w-tree T is bounded if $\pi_a(T) = T$ for some finite a. Notice that a tree T is bounded if and only if $H_{max}(T)$ is finite.

2.4. **Grafting procedure.** We will define in this section a procedure by which we add (graft) w-trees on an existing w-tree. More precisely, let $T \in \mathbb{T}$ and let $((T_i, x_i), i \in I)$ be a finite or countable family of elements of $\mathbb{T} \times T$. We define the real tree obtained by grafting the trees T_i on T at point x_i. We set $\tilde{T} = T \sqcup \left(\bigsqcup_{i\in I} T_i \backslash \{\emptyset^{T_i}\}\right)$ where the symbol $\sqcup$ means that we choose for the sets T and $(T_i)_{i\in I}$ representatives of isometry classes in $\mathbb{T}$ which are disjoint subsets of some common set and that we perform the disjoint union of all these sets. We set $\emptyset^{\tilde{T}} = \emptyset^T$. The set $\tilde{T}$ is endowed with the following metric $d^{\tilde{T}}$: if $s, t \in \tilde{T}$,

$$d^{\tilde{T}}(s,t) = \begin{cases} d^T(s,t) & \text{if } s,t \in T, \\ d^T(s,x_i) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } s \in T,\ t \in T_i\backslash\{\emptyset^{T_i}\}, \\ d^{T_i}(s,t) & \text{if } s,t \in T_i\backslash\{\emptyset^{T_i}\}, \\ d^T(x_i,x_j) + d^{T_j}(\emptyset^{T_j}, s) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } i \neq j \text{ and } s \in T_j\backslash\{\emptyset^{T_j}\},\ t \in T_i\backslash\{\emptyset^{T_i}\}. \end{cases}$$

We define the mass measure on $\tilde{T}$ by:

$$\mathbf{m}^{\tilde{T}} = \mathbf{m}^T + \sum_{i\in I} \mathbf{1}_{T_i\backslash\{\emptyset^{T_i}\}}\mathbf{m}^{T_i} + \mathbf{m}^{T_i}(\{\emptyset^{T_i}\})\delta_{x_i},$$

where δ_x is the Dirac mass at point x. It is clear that the metric space $(\tilde{T}, d^{\tilde{T}}, \emptyset^{\tilde{T}})$ is still a rooted complete real tree. However, it is not always true that $\tilde{T}$ remains locally compact, or, for that matter, that $\mathbf{m}^{\tilde{T}}$ defines a locally finite measure (on $\tilde{T}$). So, we will have to check that $(\tilde{T}, d^{\tilde{T}}, \emptyset^{\tilde{T}}, \mathbf{m}^{\tilde{T}})$ is a w-tree in the particular cases we will consider.

We will use the following notation:

$$(\tilde{T}, d^{\tilde{T}}, \emptyset^{\tilde{T}}, \mathbf{m}^{\tilde{T}}) = T \circledast_{i\in I} (T_i, x_i) \tag{6}$$

and write $\tilde{T}$ instead of $(\tilde{T}, d^{\tilde{T}}, \emptyset^{\tilde{T}}, \mathbf{m}^{\tilde{T}})$ when there is no confusion.

2.5. **Real trees coded by functions.** Lévy trees are natural generalizations of Aldous's Brownian tree, where the underlying process coding for the tree (reflected Brownian motion in Aldous's case) is replaced by a certain functional of a Lévy process, the *height process*. Le Gall and Le Jan [21] and Duquesne and Le Gall [11] showed how to generate random real trees using the excursions of a Lévy process. We shall briefly recall this construction, in order to introduce the pruning procedure on Lévy trees. Let us first work in a deterministic setting.

Let f be a continuous non-negative function defined on $[0, +\infty)$, such that $f(0) = 0$, with compact support $[0, \sigma^f]$. Let d^f be the non-negative function defined by:

$$d^f(s,t) = f(s) + f(t) - 2 \inf_{u\in[s\wedge t, s\vee t]} f(u).$$

It can be easily checked that d^f is a semi-metric on $[0, \sigma^f]$. One can define the equivalence relation associated to d^f by $s \sim t$ if and only if $d^f(s,t) = 0$. Moreover, when we consider the quotient space

$$T^f = [0, \sigma_f]_{/\sim}$$

and, noting again d^f the induced metric on T^f and rooting T^f at $\emptyset^f$, the equivalence class of 0, it can be checked that the space $(T^f, d^f, \emptyset^f)$ is a compact rooted real tree. We denote by p^f the canonical projection from $[0, \sigma^f]$ onto T^f. Notice that p^f is continuous. We define $\mathbf{m}^{T^f}$, the mass measure on T^f as the image measure on T^f of the Lebesgue measure on $\mathbb{R}_+$ by p^f. We consider the (compact) w-tree $(T^f, d^f, \emptyset^f, \mathbf{m}^f)$.

It should be noted that, if $x \in T^f$ is an equivalence class, the common value of f on all the points in this equivalence class is exactly $d^f(\emptyset, x) = H_x$. Notice that, in this setting, $H_{max}(T^f) = \|f\|_\infty$ where $\|f\|_\infty$ stands for the uniform norm of f.

We have the following elementary result (see Lemma 2.3 of [11] when dealing with the Gromov-Hausdorff metric instead of the Gromov-Hausdorff-Prokhorov metric).

Proposition 2.11. *Let f, g be two compactly supported, non-negative continuous functions with $f(0) = g(0) = 0$. Then:*

$$d_f(T^f, T^g) \leq 4\|f-g\|_\infty + |\sigma^f - \sigma^g|, \tag{7}$$

where $\sigma^f = \sup \mathrm{Supp}\, f < \infty$.

Proof. Let us consider the space $T^f \amalg_\emptyset T^g = (T^f \setminus \{\emptyset^f\}) \amalg (T^g \setminus \{\emptyset^g\} \amalg \{\emptyset\}$ (informally, the reunion of T^f and T^g joined by the root), with the distance defined by:

$$d(s,t) = \begin{cases} d^i(s,t) & \text{if } s,t \in T^i \setminus \{\emptyset^{T^i}\},\ i \in \{f,g\} \\ d^i(s, \emptyset^i) & \text{if } s \in T^i,\ t = \emptyset,\ i \in \{f,g\} \\ \inf_{x\in\mathbb{R}}\{d^f(s,p^f(x)) + d^g(t,p^g(x)) + 2\|f-g\|_\infty\}, & \text{if } s \in T^f \setminus \{\emptyset^f\},\ t \in T^g \setminus \{\emptyset^g\}. \end{cases}$$

It can be checked, as in [9], Theorem 7.3.25, that d is indeed a metric on $T^f \amalg_\emptyset T^g$. In order to bound $d_f(T^f, T^g)$, it is then sufficient to compute the relevant distances with respect to this metric space, using the obvious isometric embeddings $\Phi_{can} : T^f \hookrightarrow T^f \amalg_\emptyset T^g$ and $\Phi'_{can} : T^g \hookrightarrow T^f \amalg_\emptyset T^g$. Obviously, $d(\Phi_{can}(\emptyset^f), \Phi'_{can}(\emptyset^g)) = 0$, since we identified the roots to $\emptyset$. We have

$$d_H^{(T^f \amalg_\emptyset T^g, d)}(\Phi_{can}(T^f), \Phi'_{can}(T^g)) = \sup_{x_f \in T^f}\{\inf_{x_g \in T^g} d(x_f, x_g)\} \vee \sup_{x_g \in T^g}\{\inf_{x_f\in T^f} d(x_f,x_g)\},$$

which is equal to $2\|f-g\|_\infty$, since points $x_f = p^f(t)$ and $x_g = p^g(t)$ for $t \in \mathbb{R}$ are always at distance bounded by $2\|f-g\|_\infty$. Furthermore, if $x_f = p^f(t)$ is given, then we can always consider $x_g = p^g(t)$, which will either be a point of $T^g \setminus \{\emptyset^{T^g}\}$ or $\emptyset$. Either way, the distance is bounded by $2\|f-g\|_\infty$.

Turning to the Prokhorov distance between $(\Phi_{can})_*\mathbf{m}^f$ and $(\Phi'_{can})_*\mathbf{m}^g$, let us assume, without loss of generality, that $\sigma^f \leq \sigma^g$. If $A \subset T^f$, then we have $(p^f)^{-1}(A) \subset [0, \sigma^g]$ so that $B = p^g((p^f)^{-1}(A)) \subset T^g$. By definition, we have $\mathbf{m}^f(A) = \mathrm{Leb}((p^f)^{-1}(A)) = \mathbf{m}^g(B)$. But if $y \in B$, we can write $y = p^g(t),\ t \in (p^f)^{-1}(A)$, so that $d(x,y) = 2\|f-g\|_\infty$ with $x = p^f(t) \in A$. Thus $B \subset A^{2\|f-g\|_\infty}$. We can then write:

$$\mathbf{m}^f(A) = \mathbf{m}^g(B) \leq \mathbf{m}^g(A^{2\|f-g\|}). \tag{8}$$

If $A \subset T^g$, we can decompose

$$(p^g)^{-1}(A) = \left((p^g)^{-1}(A) \cap [0,\sigma^f]\right) \amalg \left((p^g)^{-1}(A) \cap (\sigma^f, \sigma^g]\right),$$

so that the same argument as above can be applied to the first part. This leads to

$$\begin{aligned} \mathbf{m}^g(A) &= \mathrm{Leb}((p^g)^{-1}(A) \cap [0,\sigma^f]) + \mathrm{Leb}((p^g)^{-1}(A) \cap (\sigma^f, \sigma^g]) && (9) \\ &\leq \mathbf{m}^f(A^{2\|f-g\|_\infty}) + |\sigma^f - \sigma^g|, && (10) \end{aligned}$$

which shows the inequality. $\square$

This enables us to translate all the results stated in terms of the exploration process (see the next section for the definition of the exploration process) such as the special Markov property, in analogous results on real trees. Such translations are proved in details in [11], we will omit the proofs here.

Remark 2.12. We could define the correspondence for more general functions f: lower semi-continuous functions that satisfy the intermediate values property (see [10]). In that case, the associated real tree is not even locally compact (hence not necessarily proper). But the measurability of the mapping $f \mapsto T^f$ is not clear in this general setting, that is why we only consider continuous function f here and thus will assume the Grey condition (see next Section) for Lévy trees.

2.6. **Branching mechanisms.** Let Π be a σ-finite measure on $(0,+\infty)$ such that we have $\int(1 \wedge x^2)\Pi(dx) < \infty$. We set:

$$\Pi_\theta(dr) = \mathrm{e}^{-\theta r}\ \Pi(dr). \tag{11}$$

Let Θ' be the set of $\theta \in \mathbb{R}$ such that $\int_{(1,+\infty)} \Pi_\theta(dr) < +\infty$. If $\Pi = 0$, then $\Theta' = \mathbb{R}$. We also set $\theta_\infty = \inf \Theta'$. It is obvious that $[0,+\infty) \subset \Theta'$, $\theta_\infty \leq 0$ and either $\Theta' = [\theta_\infty, +\infty)$ or $\Theta' = (\theta_\infty, +\infty)$.

Let $\alpha \in \mathbb{R}$ and $\beta \geq 0$. We consider the branching mechanism ψ associated with (α, β, Π):

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} (\mathrm{e}^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r<1\}})\Pi(dr), \quad \lambda \in \Theta'. \tag{12}$$

Notice that the function ψ is smooth and convex over $(\theta_\infty, +\infty)$. We say that ψ is conservative if for all $\varepsilon > 0$:

$$\int_{(0,\varepsilon]} \frac{du}{|\psi(u)|} = +\infty.$$

A sufficient condition for ψ to be conservative is to have $\psi'(0+) > -\infty$. This last condition is actually equivalent to $\int_{(1,\infty)} r\Pi(dr) < \infty$. We will always make the following assumption.

Assumption 1. *The function ψ is conservative and we have $\beta > 0$ or $\int_{(0,1)} \ell\Pi(d\ell) = +\infty$.*

The branching mechanism is said to be sub-critical (resp. critical, super-critical) if $\psi'(0+) > 0$ (resp. $\psi'(0+) = 0$, $\psi'(0+) < 0$). We say that ψ is (sub)critical if it is critical or sub-critical.

We introduce the following branching mechanisms ψ_θ for $\theta \in \Theta'$:

$$\psi_\theta(\lambda) = \psi(\lambda + \theta) - \psi(\theta), \quad \lambda + \theta \in \Theta'. \tag{13}$$

Let Θ^ψ be the set of $\theta \in \Theta'$ such that ψ_θ is conservative. Obviously, we have:

$$[0,+\infty) \subset \Theta^\psi \subset \Theta' \subset \Theta^\psi \cup \{\theta_\infty\}.$$

We will later on consider the following assumption.

Assumption 2. *(Grey condition) The branching mechanism is such that:*

$$\int^{+\infty} \frac{du}{\psi(u)} < \infty.$$

Let us remark that Assumption 2 implies that $\beta > 0$ or $\int_{(0,1)} r\Pi(dr) = +\infty$.

2.7. **Connections with branching processes.** Let ψ be a branching mechanism satisfying Assumption 1. A continuous state branching processes (CSBP) with branching mechanism ψ and initial mass $x > 0$ is the càdlàg $\mathbb{R}_+$-valued Markov process $(Z_a,\ a \geq 0)$ whose distribution is characterized by $Z_0 = x$ and:

$$\mathbf{E}[\exp(-\lambda Z_{a+a'})|Z_a] = \exp(-Z_a u(a', \lambda)), \quad \lambda \geq 0,$$

where $(u(a,\lambda), a \geq 0, \lambda > 0)$ is the unique non-negative solution to the integral equation:

$$\int_{u(a,\lambda)}^{\lambda} \frac{dr}{\psi(r)} = a\ ; \quad u(0,\lambda) = \lambda. \tag{14}$$

The distribution of the CSBP started at mass x will be noted $\mathbf{P}_x^\psi$. For a detailed presentation of CSBPs, we refer to the monographs [18],[19] or [22].

In this context, the conservative assumption is equivalent to the CSBP not blowing up in finite time, and Assumption 2 is equivalent to the strong extinction time, $\inf\{a; Z_a = 0\}$, being a.s. finite.

If Assumption 2 holds, then for all $h > 0$, $\mathbf{P}_x^\psi(Z_h > 0) = \exp(-xb(h))$, where $b(h) = \lim_{\lambda\to+\infty} u(h, \lambda)$. In particular $b(h)$ is such that

$$\int_{b(h)}^{\infty} \frac{dr}{\psi(r)} = h. \tag{15}$$

Let us now describe a Girsanov transform for CSBPs introduced in [3] related to the shift of the branching mechanism ψ defined by (13). Recall notation Θ^ψ and θ_∞ from the previous Section. For $\theta \in \Theta^\psi$, we consider the process $M^{\psi,\theta} = (M_a^{\psi,\theta}, a \geq 0)$ defined by:

$$M_a^{\psi,\theta} = \exp\Big(\theta x - \theta Z_a - \psi(\theta)\int_0^a Z_s ds\Big). \tag{16}$$

Theorem 2.13 (Girsanov transformation for CSBPs, [3]). *Let ψ be a branching mechanism satisfying Assumption 1. Let $(Z_a, a \geq 0)$ be a CSBP with branching mechanism ψ and let $\mathcal{F} = (\mathcal{F}_a, a \geq 0)$ be its natural filtration. Let $\theta \in \Theta^\psi$ such that either $\theta \geq 0$ or $\theta < 0$ and $\int_{(1,+\infty)} r\Pi_\theta(dr) < +\infty$. Then we have the following:*

1. *The process $M^{\psi,\theta}$ is a $\mathcal{F}$-martingale under $\mathbf{P}_x^\psi$.*
2. *Let $a, x \geq 0$. On $\mathcal{F}_a$, the probability measure $\mathbf{P}_x^{\psi_\theta}$ is absolutely continuous w.r.t. $\mathbf{P}_x^\psi$, and*

$$\frac{d\mathbf{P}_{x\,|\mathcal{F}_a}^{\psi_\theta}}{d\mathbf{P}_{x\,|\mathcal{F}_a}^{\psi}} = M_a^{\psi,\theta}.$$

2.8. **The height process.** Let $(X_t, t \geq 0)$ be a Lévy process with Laplace exponent ψ satisfying Assumption 1. This assumption implies that a.s. the paths of X have infinite total variation over any non-trivial interval. The distribution of the Lévy process will be noted $\mathbb{P}^\psi(dX)$. It is a probability measure on the Skorokhod space of real-valued càdlàg processes. For the remainder of this section, **we will assume that ψ is (sub)critical**.

For $t \geq 0$, let us write $\hat{X}^{(t)}$ for the time-returned process:

$$\forall\ 0 \leq s < t,\ \hat{X}_s^{(t)} = X_t - X_{(t-s)_-}$$

and $\hat{X}_t^{(t)} = X_t$. Then $(\hat{X}_s^{(t)}, 0 \leq s \leq t)$ has same distribution as the process $(X_s, 0 \leq s \leq t)$. We will also write $\hat{S}_s^{(t)} = \sup_{[0,s]} \hat{X}_r^{(t)}$ for the supremum process of $\hat{X}^{(t)}$.

Proposition 2.14 (The height process, [10]). *There exists a lower semi-continuous process $H = (H_t, t \geq 0)$ taking values in $[0, +\infty]$, with the intermediate values property, which is a local time at 0, at time t, of the process $\hat{X}^{(t)} - \hat{S}^{(t)}$, such that the following convergence holds in probability:*

$$H_t = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \mathbf{1}_{\{I_s^t \leq X_s \leq I_s^t + \varepsilon\}} ds$$

where $I_s^t = \inf_{s\leq r\leq t} X_r$. Furthermore, if Assumption 2 holds, then the process H admits a continuous modification.

From now on, we always assume that Assumptions 1 and 2 hold, and we always work with this continuous version of H. The process H is called the height process.

For $x > 0$, we consider the stopping time $\tau_x = \inf\Big\{t \geq 0,\ I_t \leq -x\Big\}$, where $I_t = I_0^t$ is the infimum process of X. We denote by $\mathbb{P}_x^\psi(dH)$ the distribution of the stopped height process $(H_{t\wedge\tau_x}, t \geq 0)$ under $\mathbb{P}^\psi$, defined on the space $\mathcal{C}_+([0, +\infty))$ of non-negative continuous functions on $[0, +\infty)$. The (sub)criticality of the branching mechanism entails $\tau_x < \infty$ $\mathbb{P}^\psi$-a.s., so that under $\mathbb{P}_x^\psi(dH)$, a.s. the height process has compact support.

2.9. **The excursion measure.** The height process is not a Markov process, but it has the same zero sets as $X - I$ (see [10], Paragraph 1.3.1), so we can develop an excursion theory based on the latter. By standard fluctuation theory, it is easy to see that 0 is a regular point for $X - I$ and that $-I$ is a local time of $X - I$ at 0. We denote by $\mathbb{N}^\psi$ the associated excursion measure. As such, $\mathbb{N}^\psi$ is a σ-finite measure. Under $\mathbb{P}_x^\psi$ or $\mathbb{N}^\psi$, we set:

$$\sigma(H) = \int_0^\infty \mathbf{1}_{\{H_t \neq 0\}} dt.$$

When there is no risk of confusion, we will write σ for $\sigma(H)$. Notice that, under $\mathbb{P}_x^\psi$, $\sigma = \tau_x$ and that under $\mathbb{N}^\psi$, σ represents the lifetime of the excursion. Abusing notations, we will write $\mathbb{P}_x^\psi(dH)$ and $\mathbb{N}^\psi[dH]$ for the distribution of H under $\mathbb{P}_x^\psi$ or $\mathbb{N}^\psi$. Let us also recall the Poissonian decomposition of the measure $\mathbb{P}_x^\psi$. Under $\mathbb{P}_x^\psi$, let $(a_j, b_j)_{j\in J}$ be the excursion intervals of $X - I$ away from 0. Those are also the excursion intervals of the height process away from 0. For $j \in J$, we shall denote by $H^{(j)} : [0, \infty) \to \mathbb{R}_+$ the corresponding excursion, that is

$$H_t^{(j)} = H_{(a_j+t)\wedge b_j}, \quad t \geq 0.$$

Proposition 2.15 ([11])**.** *Under $\mathbb{P}_x^\psi$, the random point measure $\mathcal{N} = \sum_{j\in J} \delta_{H^{(j)}}(dH)$ is a Poisson point measure with intensity $x\mathbb{N}^\psi[dH]$.*

2.10. **Local times of the height process.**

Proposition 2.16 ([10], Formula (36))**.** *Under $\mathbb{N}^\psi$, there exists a jointly measurable process $(L_s^a, a \geq 0, s \geq 0)$ which is continuous and non-decreasing in the variable s such that,*

$$\forall s \geq 0,\ L_s^0 = 0$$

and for every $t \geq 0$, for every $\delta > 0$ and every $a > 0$

$$\lim_{\varepsilon\to 0} \mathbb{N}^\psi \left[\mathbf{1}_{\{\sup H > \delta\}} \sup_{0\leq s\leq t\wedge\sigma} \left| \varepsilon^{-1} \int_0^s \mathbf{1}_{\{a < H_r \leq a+\varepsilon\}}\, dr - L_s^a \right| \right] = 0.$$

Moreover, by Lemma 3.3. of [11], the process $(L_\sigma^a,\ a \geq 0)$ has a càdlàg modification under $\mathbb{N}^\psi$ with no fixed discontinuities.

2.11. **(Sub)critical Lévy trees.** Let ψ be a (sub)critical branching mechanism satisfying Assumptions 1 and 2. Let H be the height process defined under $\mathbb{P}_x^\psi$ or $\mathbb{N}^\psi$. We consider the so-called Lévy tree $\mathcal{T}^H$ which is the random w-tree coded by the function H, see Section 2.5. Notice that we are indeed within the framework of proper real trees, since Assumption 2 entails compactness of $\mathcal{T}^H$. When there is no confusion, we shall write $\mathcal{T}$ for $\mathcal{T}_H$. Abusing notations, we will write $\mathbb{P}_x^\psi(d\mathcal{T})$ and $\mathbb{N}^\psi[d\mathcal{T}]$ for the distribution on $\mathbb{T}$ of $\mathcal{T} = \mathcal{T}^H$ under $\mathbb{P}_x^\psi(dH)$ or $\mathbb{N}^\psi[dH]$. By construction, under $\mathbb{P}_x^\psi$ or under $\mathbb{N}^\psi$, we have that the total mass of the mass measure on $\mathcal{T}$ is given by:

$$\mathbf{m}^{\mathcal{T}}(\mathcal{T}) = \sigma. \tag{17}$$

Proposition 2.15 enables us to view the measure $\mathbb{N}^\psi[d\mathcal{T}]$ as describing a single Lévy tree. Thus, we will mostly work under this excursion measure, which is the distribution of the (isometry class of the) w-tree $\mathcal{T}$ described by the height process under $\mathbb{N}^\psi$. In order to state the branching property of a Lévy tree, we must first define a local time at level a on the tree. Let $(\mathcal{T}^{i,\circ}, i \in I)$ be the trees that were cut off by cutting at level a, namely the connected components of the set $\mathcal{T} \setminus \pi_a(\mathcal{T})$. If $i \in I$, then all the points in $\mathcal{T}^{i,\circ}$ have the same MRCA x_i in $\mathcal{T}$ which is precisely the point where the tree was cut off. We consider the compact tree $\mathcal{T}^i = \mathcal{T}^{i,\circ} \cup \{x_i\}$ with the root x_i, the metric $d^{\mathcal{T}^i}$, which is the metric $d^{\mathcal{T}}$ restricted to $\mathcal{T}^i$, and the mass measure $\mathbf{m}^{\mathcal{T}^i}$, which is the mass measure $\mathbf{m}^{\mathcal{T}}$ restricted to $\mathcal{T}^i$. Then $(\mathcal{T}^i, d^{\mathcal{T}^i}, x_i, \mathbf{m}^{\mathcal{T}^i})$ is a w-tree. Let

$$\mathcal{N}_a^{\mathcal{T}}(dx, d\mathcal{T}') = \sum_{i\in I} \delta_{(x_i,\mathcal{T}^i)}(dx, d\mathcal{T}') \tag{18}$$

be the point measure on $\mathcal{T}(a) \times \mathbb{T}$ taking account of the cutting points as well as the trees cut away. The following theorem gives the structure of the decomposition we just described. From excursion theory, we deduce that $b(h) = \mathbb{N}^{\psi}[H_{max}(\mathcal{T}) > h]$, where $b(h)$ solves (15). An easy extension of [11] from real trees to w-trees gives the following result.

Theorem 2.17 ([11]). *For every fixed $a \geq 0$ and for $\mathbb{N}^{\psi}$-a.e. $\mathcal{T} \in \mathbb{T}$, we can define a finite measure ℓ^a on $\mathcal{T}$ such that*

- *$\ell^0 = 0$ and $\mathbb{N}^{\psi}$-a.e., ℓ^a is supported on $\mathcal{T}(a)$,*
- *$\mathbb{N}^{\psi}$-a.e., $\{\ell^a \neq 0\} = \{H_{max}(\mathcal{T}) > a\}$,*
- *We have, $\mathbb{N}^{\psi}$-a.e. for every bounded continuous function φ on $\mathcal{T}$,*

$$\langle \ell^a, \varphi\rangle = \lim_{\varepsilon\downarrow 0} \frac{1}{b(\varepsilon)} \int \varphi(x)\mathbf{1}_{\{h(\mathcal{T}')\geq\varepsilon\}} \mathcal{N}_a^{\mathcal{T}}(dx, d\mathcal{T}') \tag{19}$$

$$= \lim_{\varepsilon\downarrow 0} \frac{1}{b(\varepsilon)} \int \varphi(x)\mathbf{1}_{\{h(\mathcal{T}')\geq\varepsilon\}} \mathcal{N}_{a-\varepsilon}^{\mathcal{T}}(dx, d\mathcal{T}'),\ \textit{if } a > 0. \tag{20}$$

Furthermore, we have the branching property: for every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^{\mathcal{T}}(dx, d\mathcal{T}')$ under $\mathbb{N}^{\psi}[d\mathcal{T}|H_{max}(\mathcal{T}) > a]$, given $\pi_a(\mathcal{T})$, is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(dx)\mathbb{N}^{\psi}[d\mathcal{T}']$.

Eventually, one can choose a modification of the collection $(\ell^a, a \geq 0)$ in such a way that the mapping $a \mapsto \ell^a$ is $\mathbb{N}^{\psi}$-a.e. càdlàg for the weak topology on finite measures on $\mathcal{T}$. We have then:

$$\inf\{a > 0; \ell^a = 0\} = \sup\{a \geq 0; \ell^a \neq 0\} = H_{max}(\mathcal{T}) \quad \mathbb{N}^{\psi}\textit{-a.e.}$$

The measure ℓ^a will be called the *local time* measure of $\mathcal{T}$ at level a. In the case of Lévy trees, it can also be defined as the image of the measure $d_s L_s^a(H)$ by the canonical projection p_H (see [10]), so the above statement is in fact the translation of the excursion theory of the height process in terms of real trees. This definition shows that the local time is a function of the tree $\mathcal{T}$ and does not depend on the choice of the coding height function. It should be noted that Equation (20) implies that ℓ^a is measurable with respect to the σ-algebra generated by $\pi_a(\mathcal{T})$. According to [11], we also have the following representation of the mass measure $\mathbf{m}^{\mathcal{T}}$:

$$\mathbf{m}^{\mathcal{T}}(dx) = \int_0^{\infty} \ell^a(dx)da. \tag{21}$$

The next theorem, also from [11], relates the discontinuities of the process $(\ell^a, a \geq 0)$ to the infinite nodes in the tree. Recall $\mathrm{Br}_{\infty}(\mathcal{T})$ denotes the set of infinite nodes in the Lévy tree $\mathcal{T}$.

Theorem 2.18 ([11]). *The set $\{d(\emptyset, x),\ x \in \mathrm{Br}_{\infty}(\mathcal{T})\}$ coincides $\mathbb{N}^{\psi}$-a.e. with the set of discontinuity times of the mapping $a \mapsto \ell^a$. Moreover, $\mathbb{N}^{\psi}$-a.e., for every such discontinuity time b, there is a unique $x_b \in \mathrm{Br}_{\infty}(\mathcal{T}) \cap \mathcal{T}(b)$, and*

$$\ell^b = \ell^{b-} + \Delta_b \delta_{x_b},$$

where $\Delta_b > 0$ is called mass *of the node x_b and can be obtained by the approximation*

$$\Delta_b = \lim_{\varepsilon\to 0} \frac{1}{b(\varepsilon)} n(x_b, \varepsilon), \tag{22}$$

where $n(x_b, \varepsilon) = \int \mathbf{1}_{\{x=x_b\}}(x)\mathbf{1}_{\{H_{max}(\mathcal{T}')>\varepsilon\}}(\mathcal{T}')\mathcal{N}_b^{\mathcal{T}}(dx, d\mathcal{T}')$ is the number of sub-trees originating from x_b with height larger than ε.

2.12. **Decomposition of the Lévy tree.** We will frequently use the following notation for the following measure on $\mathbb{T}$:

$$\mathbf{N}^\psi[\mathcal{T} \in \bullet] = 2\beta \mathbb{N}^\psi[\mathcal{T} \in \bullet] + \int_{(0,+\infty)} r\Pi(dr)\, \mathbb{P}_r^\psi[\mathcal{T} \in \bullet]. \tag{23}$$

where ψ is given by (12).

The decomposition of a (sub)critical Lévy tree $\mathcal{T}$ according to a spine $[\![\emptyset, x]\!]$, where $x \in \mathcal{T}$ is a leaf picked at random at level $a > 0$, that is according to the local time $\ell^a(dx)$, is given in Theorem 4.5 in [11]. Then by integrating with respect to a, we get the decomposition of $\mathcal{T}$ according to a spine $[\![\emptyset, x]\!]$, where $x \in \mathcal{T}$ is a leaf picked at random on $\mathcal{T}$, that is according to the mass measure $\mathbf{m}^\mathcal{T}$. Therefore, we will state this decomposition without proof.

Let $x \in \mathcal{T}$ and $\{x_i, i \in I_x\}$ the set $\mathrm{Br}(\mathcal{T}) \cap [\![\emptyset, x]\!]$ of branching point on the spine $[\![\emptyset, x]\!]$. For $i \in I_x$, we set:

$$\mathcal{T}^i = \mathcal{T} \setminus \left(\mathcal{T}^{(x,x_i)} \cup \mathcal{T}^{(\emptyset,x_i)} \} \right),$$

where $\mathcal{T}^{(y,x_i)}$ is the connected component of $\mathcal{T} \setminus \{x_i\}$ containing y. We let x_i be the root of $\mathcal{T}^i$. The metric and measure on $\mathcal{T}^i$ are respectively the restriction of $d^\mathcal{T}$ to $\mathcal{T}^i$ and the restriction of $\mathbf{m}^\mathcal{T}$ to $\mathcal{T}^i \backslash \{x_i\}$. By construction, if x is a leaf, we have:

$$\mathcal{T} = [\![\emptyset, x]\!] \circledast_{i \in I_x} (\mathcal{T}^i, x_i),$$

where $[\![\emptyset, x]\!]$ is a w-tree with root $\emptyset$, metric and mass measure the restrictions of $d^\mathcal{T}$ and $\mathbf{m}^\mathcal{T}$ to $[\![\emptyset, x]\!]$.

We consider the point measure on $[0, H_x] \times \mathbb{T}$ defined by:

$$\mathcal{M}_x = \sum_{i \in i_x} \delta_{(H_{x_i}, \mathcal{T}^i)}.$$

Theorem 2.19 ([11]). *Assume ψ is (sub)critical and satisfies Assumptions 1 and 2. We have for any non-negative measurable function F defined on $[0, +\infty) \times \mathbb{T}$:*

$$\mathbb{N}^\psi \left[\int \mathbf{m}^\mathcal{T}(dx) F(H_x, \mathcal{M}_x) \right] = \int_0^\infty da\ \mathrm{e}^{-\psi'(0)a}\ \mathbb{E}\left[F\left(a, \sum_{i \in I} \mathbf{1}_{\{z_i \le a\}} \delta_{(z_i, \bar{\mathcal{T}}^i)} \right) \right],$$

where under $\mathbb{E}$, $\sum_{i \in I} \delta_{(z_i, \bar{\mathcal{T}}^i)}(dz, dT)$ is a Poisson point measure on $[0, +\infty) \times \mathbb{T}$ with intensity $dz\, \mathbf{N}^\psi[dT]$.

2.13. **CSBP process in the Lévy trees.** Lévy trees give a genealogical structure for CSBPs, which is precised in the next Theorem. We consider the process $\mathcal{Z} = (\mathcal{Z}_a, a \ge 0)$ defined by:

$$\mathcal{Z}_a = \langle \ell^a, 1 \rangle.$$

If needed we will write $\mathcal{Z}_a(\mathcal{T})$ to emphasize that $\mathcal{Z}_a$ corresponds to the tree $\mathcal{T}$.

Theorem 2.20 (CSBP in Lévy trees, [10] and [11]). *Let ψ be a (sub)critical branching mechanism satisfying Assumptions 1 and 2, and let $x > 0$. The process $\mathcal{Z}$ under $\mathbb{P}_x^\psi$ is distributed as the CSBP Z under $\mathbf{P}_x^\psi$.*

Remark 2.21. This theorem can be stated in terms of the height process without Assumption 2.

2.14. **Super-critical Lévy trees.** Let us now briefly recall the construction from [3] for super-critical Lévy trees using a Girsanov transformation similar to the one used for CSBPs, see Theorem 2.13.

Let ψ be a super-critical branching mechanism satisfying Assumptions 1 and 2. Let θ^* be the unique positive root of ψ'. The branching mechanism ψ_θ is sub-critical if $\theta > \theta^*$, critical if $\theta = \theta^*$ and super-critical otherwise. We consider the filtration $\mathcal{H} = (\mathcal{H}_a,\ a \ge 0)$, where $\mathcal{H}_a$ is the σ-field generated by the random variable $\pi_a(\mathcal{T})$ and the $\mathbb{P}_x^{\psi_{\theta^*}}$-negligible sets. For $\theta \geq \theta^*$, we define the process $M^{\psi,\theta} = (M_a^{\psi,\theta}, a \geq 0)$ with:

$$M_a^{\psi,\theta} = \exp\Big(\theta x - \theta \mathcal{Z}_a - \psi(\theta) \int_0^a \mathcal{Z}_s ds\Big)$$

By absolute continuity of the measures $\mathbb{P}_x^{\psi_\theta}$ (resp. $\mathbb{N}^{\psi_\theta}$) with respect to $\mathbb{P}_x^{\psi_{\theta^*}}$ (resp. $\mathbb{N}^{\psi_{\theta^*}}$), all the processes $M^{\psi_\theta,-\theta}$ for $\theta > \theta^*$ are $\mathcal{H}$-adapted. Moreover, all these processes are $\mathcal{H}$-martingales (see [3] for the proof). Theorem 2.17 shows that $M^{\psi_{\theta^*},-\theta^*}$ is $\mathcal{H}$-adapted. Let us now define the ψ-Lévy tree, cut at level a by the following Girsanov transformation.

Definition 2.22. *Let ψ be a super-critical branching mechanism satisfying Assumptions 1 and 2. Let $\theta \geq \theta^*$. For $a \geq 0$, we define the distribution $\mathbb{P}_x^{\psi,a}$ (resp. $\mathbb{N}^{\psi,a}$) by: if F is a non-negative, measurable functional defined on $\mathbb{T}$,*

$$\mathbb{E}_x^{\psi,a}[F(\mathcal{T})] = \mathbb{E}_x^{\psi_\theta}\Big[M_a^{\psi_\theta,-\theta} F(\pi_a(\mathcal{T}))\Big], \tag{24}$$

$$\mathbb{N}^{\psi,a}[F(\mathcal{T})] = \mathbb{N}^{\psi_\theta}\Big[\exp\Big(\theta \mathcal{Z}_a + \psi(\theta) \int_0^a \mathcal{Z}_s ds) F(\pi_a(\mathcal{T})\Big)\Big]. \tag{25}$$

It can be checked that the definition of $\mathbb{P}_x^{\psi,a}$ (and of $\mathbb{N}^{\psi,a}$) does not depend on $\theta \geq \theta^*$.

The probability measures $\mathbb{P}_x^{\psi,a}$ satisfy a consistence property, allowing us to define the super-critical Lévy tree in the following way.

Theorem 2.23. *There exists a probability measure $\mathbb{P}_x^\psi$ (resp. a σ-finite measure $\mathbb{N}^\psi$) on $\mathbb{T}$ such that for $a > 0$, we have, if F is a measurable non-negative functional on $\mathbb{T}$,*

$$\mathbb{E}_x^\psi[F(\pi_a(\mathcal{T}))] = \mathbb{E}_x^{\psi,a}[F(\mathcal{T})],$$

the same being true under $\mathbb{N}^\psi$.

The w-tree $\mathcal{T}$ under $\mathbb{P}_x^\psi$ or $\mathbb{N}^\psi$ is called a ψ-Lévy w-tree or simply a Lévy tree.

Proof. For $n \geq 1,\ 0 < a_1 < ... < a_n$, we define a probability measure on $\mathbb{T}^n$ by:

$$\mathbb{P}_x^{\psi,a_1,...,a_n}(\mathcal{T}_1 \in A_1, ..., \mathcal{T}_n \in A_n) = \mathbb{P}_x^{\psi,a_n}(\mathcal{T} \in A_n, \pi_{a_{n-1}}(\mathcal{T}) \in A_{n-1}, ..., \pi_{a_1}(\mathcal{T}) \in A_1)$$

if $A_1, ..., A_n$ are Borel subsets of $\mathbb{T}$. The probability measures $\mathbb{P}_x^{\psi,a_1,...,a_n}$ for $n \geq 1,\ 0 < a_1 < ... < a_n$ then form a projective family. This is a consequence of the martingale property of $M^{\psi_\theta,-\theta}$ and the fact that the projectors π_a satisfy the obvious compatibility relation $\pi_b \circ \pi_a = \pi_b$ if $0 < b < a$.

By the Daniell-Kolmogorov theorem, there exists a probability measure $\tilde{\mathbb{P}}_x^\psi$ on the product space $\mathbb{T}^{\mathbb{R}_+}$ such that the finite-dimensional distributions of a $\tilde{\mathbb{P}}_x^\psi$-distributed family are described by the measures defined above. It is easy to construct a version of a $\tilde{\mathbb{P}}_x^\psi$-distributed process that is a.s. increasing. Indeed, almost all sample paths of a $\tilde{\mathbb{P}}_x^\psi$-distributed process are increasing when restricted to rational numbers. We can then define a w-tree $\mathcal{T}^a$ for any $a > 0$ by considering a decreasing sequence of rational numbers $a_n \downarrow a$ and defining $\mathcal{T}^a = \cap_{n\geq 1} \mathcal{T}^{a_n}$. Notice that $\mathcal{T}^a$ is closed for all $a \in \mathbb{R}_+$. It is easy to check that the finite-dimensional distributions of this new process are unchanged by this procedure. Let us then consider $\mathcal{T} = \cup_{a>0} \mathcal{T}^a$, endowed with the obvious metric $d^{\mathcal{T}}$ and mass measure $\mathbf{m}$. It is clear that $\mathcal{T}$ is a real tree, rooted at the common root of the $\mathcal{T}^a$. All the $\mathcal{T}^a$ are compact, so that $\mathcal{T}$ is locally compact and complete. The measure $\mathbf{m}$ is locally finite since all the $\mathbf{m}^{\mathcal{T}^a}$ are finite measures. Therefore, $\mathcal{T}$ is a.s. a w-tree. Then, if we define $\mathbb{P}_x^\psi$ to be the distribution of $\mathcal{T}$, the conclusion follows. Similar arguments hold under $\mathbb{N}^\psi$. $\square$

Remark 2.24. Another definition of super-critical Lévy trees was given by Duquesne and Winkel [12],[13]: they consider increasing families of Galton-Watson trees with exponential edge lengths which satisfy a certain hereditary property (such as uniform Bernoulli coloring of the leaves). Lévy trees are then defined to be the Gromov-Hausdorff limits of these processes. Another approach via backbone decompositions is given in [8].

All the definitions we made for sub-critical Lévy trees then carry over to the super-critical case. In particular, the level set measure ℓ^a, which is $\pi_a(\mathcal{T})$-measurable, can be defined using the Girsanov formula. Thanks to Theorem 2.13, it is easy to show that the mass process $(\mathcal{Z}_a = \langle \ell^a, 1\rangle,\ a \geq 0)$ is under $\mathbb{P}_x^\psi$ a CSBP with branching mechanism ψ. In particular, with u defined in (14) and b by (15), we have:

$$\mathbb{N}^\psi\left[1-\mathrm{e}^{-\lambda \mathcal{Z}_a}\right] = u(a,\lambda) \quad \text{and} \quad \mathbb{N}^\psi\left[H_{max}(\mathcal{T}) > a\right] = \mathbb{N}^\psi\left[\mathcal{Z}_a > 0\right] = b(a). \tag{26}$$

Notice that b is finite only under Assumption 2. We set:

$$\sigma = \int_0^{+\infty} \mathcal{Z}_a \, da = \mathbf{m}^{\mathcal{T}}(\mathcal{T}) \tag{27}$$

for the total mass of the Lévy tree $\mathcal{T}$. Notice this is consistent with (21) and (17) which are defined for (sub)critical Lévy trees. Thanks to (27), notice that σ is distributed as the total population size of a CSBP with branching mechanism ψ. In particular, its Laplace transform is given for $\lambda > 0$ by:

$$\mathbb{N}^\psi[1-\mathrm{e}^{-\lambda\sigma}] = \psi^{-1}(\lambda) \tag{28}$$

where ψ^{-1} is the right-continuous inverse of ψ. Notice that $\mathbb{N}^\psi[\sigma < +\infty] = \psi^{-1}(0) \geq 0$. Let us state the following Theorem, from [3], which sums up the situation.

Theorem 2.25 ([3])**.** *Let ψ be a branching mechanism satisfying Assumptions 1 and 2, and let $q > 0$ be such that $\psi(q) \geq 0$. Then, the probability measure $\mathbb{P}_x^{\psi_q}$ on $\mathbb{T}$ is absolutely continuous w.r.t. $\mathbb{P}_x^\psi$, with*

$$\frac{d\mathbb{P}_x^{\psi_q}}{d\mathbb{P}_x^\psi} = M_\infty^{\psi,q} = \mathrm{e}^{qx-\psi(q)\sigma}\, \mathbf{1}_{\{\sigma<+\infty\}}. \tag{29}$$

Similarly, the excursion measure $\mathbb{N}^{\psi_q}$ on $\mathbb{T}$ is absolutely continuous w.r.t. $\mathbb{N}^\psi$ and we have

$$\frac{d\mathbb{N}^{\psi_q}}{d\mathbb{N}^\psi} = \mathrm{e}^{-\psi(q)\sigma}\, \mathbf{1}_{\{\sigma<+\infty\}}. \tag{30}$$

If $\theta \in \Theta^\psi$, let $\bar\theta$ be the unique positive real number such that:

$$\psi(\theta) = \psi(\bar\theta). \tag{31}$$

We also have $\bar\theta = \psi^{-1} \circ \psi(\theta)$. In particular, $\bar\theta = \theta$ when ψ_θ is (sub)critical. When applying Girsanov formula (30) to $q = \bar\theta$, we get the following remarkable Corollary, due to the fact that $\psi_\theta(\bar\theta) = 0$.

Corollary 2.26. *Let ψ be a critical branching mechanism, satisfying Assumptions 1 and 2, and $\theta \in \Theta^\psi$ with $\theta < 0$. Let F be a non-negative measurable functional defined on $\mathbb{T}$. We have:*

$$\begin{aligned} \mathrm{e}^{(\bar\theta-\theta)x}\ \mathbb{E}_x^{\psi_\theta}[F(\mathcal{T})\mathbf{1}_{\{\sigma<+\infty\}}] &= \mathbb{E}_x^{\psi_{\bar\theta}}[F(\mathcal{T})], \\ \mathbb{N}^{\psi_\theta}[F(\mathcal{T})\mathbf{1}_{\{\sigma<+\infty\}}] &= \mathbb{N}^{\psi_{\bar\theta}}[F(\mathcal{T})]. \end{aligned} \tag{32}$$

We deduce from Proposition 2.15 and Theorem 2.23 that $\mathcal{N}_0^{\mathcal{T}}(dx, d\mathcal{T}')$ defined by (18) with $a = 0$ is under $\mathbb{P}_x^\psi(d\mathcal{T})$ a Poisson point measure on $\{\emptyset\} \times \mathbb{T}$ with intensity $\sigma\delta_\emptyset(dx)\mathbb{N}^\psi[d\mathcal{T}']$. Then we deduce from (24), with $F = 1$, that for $\theta \geq \theta^*$:

$$\mathbb{N}^{\psi_\theta}\left[1-\exp\left(\theta\mathcal{Z}_a + \psi(\theta)\int_0^a \mathcal{Z}_s ds\right)\right] = -\theta. \tag{33}$$

2.15. **Pruning Lévy trees.** We recall the construction from [5] on the pruning of Lévy trees. Let $\mathcal{T}$ be a random Lévy w-tree under $\mathbb{P}^\psi_x$ (or under $\mathbb{N}^\psi$), with ψ conservative. Let

$$m^{(\mathrm{ske})}(dx, d\theta) = \sum_{i \in I^{\mathrm{ske}}} \delta_{(x_i, \theta_i)}(dx, d\theta)$$

be, conditionally on $\mathcal{T}$, a Poisson point measure on $\mathcal{T} \times \mathbb{R}_+$ with intensity $2\beta l^{\mathcal{T}}(dx)d\theta$. Since there is a.s. a countable number of branching points (which have $l^{\mathcal{T}}$-measure 0), the atoms of this measure are distributed on $\mathcal{T} \setminus (\mathrm{Br}(\mathcal{T}) \cup \mathrm{Lf}(\mathcal{T}))$.

If $\Pi = 0$, we have $\mathrm{Br}_\infty(\mathcal{T}) = \emptyset$ a.s. whereas if $\Pi(\mathbb{R}_+) = \infty$, $\mathrm{Br}_\infty(\mathcal{T})$ is a.s. a countable dense subset of $\mathcal{T}$. If the latter condition holds, we consider, conditionally on $\mathcal{T}$, a Poisson point measure

$$m^{(\mathrm{nod})}(dx, d\theta) = \sum_{i \in I^{\mathrm{nod}}} \delta_{(x_i, \theta_i)}(dx, d\theta)$$

on $\mathcal{T} \times \mathbb{R}_+$ with intensity

$$\sum_{y \in \mathrm{Br}_\infty(\mathcal{T})} \Delta_y \delta_y(dx)\, d\theta$$

where Δ_x is the mass of the node x, defined by (22). Hence, if $\theta > 0$, a node $x \in \mathrm{Br}_\infty(\mathcal{T})$ is an atom of $m^{(\mathrm{nod})}(dx, [0, \theta])$ with probability $1 - \exp(-\theta \Delta_x)$. The set

$$\{x_i,\ i \in I^{\mathrm{nod}}\} = \Big\{x \in \mathcal{T},\ m^{(\mathrm{nod})}\big(\{x\} \times \mathbb{R}_+\big) > 0\Big\}$$

of marked branching points corresponds $\mathbb{P}^\psi_x$-a.s or $\mathbb{N}^\psi$-a.e. to $\mathrm{Br}_\infty(\mathcal{T})$. For $i \in I^{\mathrm{nod}}$, we set

$$\theta_i = \inf \Big\{\theta > 0,\ m^{(\mathrm{nod})}\big(\{x_i\} \times [0, \theta]\big) > 0\Big\}$$

the first mark on x_i (which is positive), and we set

$$\{\theta_j,\ j \in J^{\mathrm{nod}}_i\} = \Big\{\theta > \theta_i,\ m^{(\mathrm{nod})}\big(\{x_i\} \times \{\theta\}\big) > 0\Big\}$$

so that we can write

$$m^{(\mathrm{nod})}(dx, d\theta) = \sum_{i \in I^{\mathrm{nod}}} \delta_{x_i}(dx) \left(\delta_{\theta_i}(d\theta) + \sum_{j \in J^{\mathrm{nod}}_i} \delta_{\theta_j}(d\theta) \right).$$

We set the measure of marks:

$$\mathcal{M}(dx, d\theta) = m^{(\mathrm{ske})}(dx, d\theta) + m^{(\mathrm{nod})}(dx, d\theta), \tag{34}$$

and consider the family of w-trees $\Lambda(\mathcal{T}, \mathcal{M}) = (\Lambda_\theta(\mathcal{T}, \mathcal{M}), \theta \geq 0)$, where the θ-pruned w-tree is defined by:

$$\Lambda_\theta(\mathcal{T}, \mathcal{M}) = \{x \in \mathcal{T},\ \mathcal{M}(\llbracket \emptyset, x \llbracket \times [0, \theta]) = 0\},$$

rooted at $\emptyset^{\Lambda_\theta(\mathcal{T}, \mathcal{M})} = \emptyset^{\mathcal{T}}$, and the metric $d^{\Lambda_\theta(\mathcal{T}, \mathcal{M})}$ and the mass measure $\mathbf{m}^{\Lambda_\theta(\mathcal{T}, \mathcal{M})}$ are the restrictions of $d^{\mathcal{T}}$ and $\mathbf{m}^{\mathcal{T}}$ to $\Lambda_\theta(\mathcal{T}, \mathcal{M})$. In particular, we have $\Lambda_0(\mathcal{T}, \mathcal{M}) = \mathcal{T}$. The family of w-trees $\Lambda(\mathcal{T}, \mathcal{M})$ is a non-increasing family of real trees, in a sense that $\Lambda_{\theta'}(\mathcal{T}, \mathcal{M}) \supset \Lambda_\theta(\mathcal{T}, \mathcal{M})$ for $0 \leq \theta' \leq \theta$, see Figure 1. In particular, we have that the pruning operators satisfy a cocycle property, for $\theta_1 \geq 0$ and $\theta_2 \geq 0$:

$$\Lambda_{\theta_2}\big(\Lambda_{\theta_1}(\mathcal{T}, \mathcal{M}), \mathcal{M}_{\theta_1}\big) = \Lambda_{\theta_2 + \theta_1}(\mathcal{T}, \mathcal{M}),$$

where $\mathcal{M}_\theta(A \times [0, q]) = \mathcal{M}(A \times [\theta, \theta + q])$. Abusing notation, we write $\mathbb{N}^\psi(d\mathcal{T}, d\mathcal{M})$ for the distribution of the pair $(\mathcal{T}, \mathcal{M})$ when $\mathcal{T}$ is distributed according to $\mathbb{N}^\psi(d\mathcal{T})$ and conditionally on $\mathcal{T}$, $\mathcal{M}$ is distributed as described above.

The following result can be deduced from [3].

Theorem 2.27. *Let ψ be a branching mechanism satisfying Assumptions 1 and 2. There exists a non-increasing $\mathbb{T}$-valued Markov process $(\mathcal{T}_\theta, \theta \in \Theta^\psi)$ such that for all $q \in \Theta^\psi$, the process $(\mathcal{T}_{\theta+q}, \theta \geq 0)$ is distributed as $\Lambda(\mathcal{T}, \mathcal{M})$ under $\mathbb{N}^{\psi_q}[d\mathcal{T}, d\mathcal{M}]$.*

In particular, this Theorem implies that $\mathcal{T}_\theta$ is distributed as $\mathbb{N}^{\psi_\theta}$ for $\theta \in \Theta^\psi$ and that for $\theta_0 \geq 0$, under $\mathbb{N}^\psi$, the process of pruned trees $(\Lambda_{\theta_0+\theta}(\mathcal{T}), \theta \geq 0)$ has the same distribution as $(\Lambda_\theta(\mathcal{T}), \theta \geq 0)$ under $\mathbb{N}^{\psi_{\theta_0}}[d\mathcal{T}]$.

We want to study the time-reversed process $(\mathcal{T}_{-\theta}, \theta \in -\Theta^\psi)$, which can be seen as a growth process. This process grows by attaching sub-trees at a random point, rather than slowly growing uniformly along the branches. We recall some results from [3] on the growth process. From now on, we will assume in this Section that the **branching mechanism ψ is critical**, so that ψ_θ is sub-critical iff $\theta > 0$ and super-critical iff $\theta < 0$.

FIGURE 1. The pruning process, starting from explosion time A defined in (36).

We will use the following notation for the total mass of the tree $\mathcal{T}_\theta$ at time $\theta \in \Theta^\psi$:

$$\sigma_\theta = \mathbf{m}^{\mathcal{T}_\theta}(\mathcal{T}_\theta). \tag{35}$$

The total mass process $(\sigma_\theta, \theta \in \Theta^\psi)$ is a pure-jump process taking values in $(0, +\infty]$.

Lemma 2.28 ([3]). *Assume ψ is critical and satisfies Assumptions 1 and 2. If $0 \leq \theta_2 < \theta_1$, then we have:*

$$\mathbb{N}^\psi[\sigma_{\theta_2} | \mathcal{T}_{\theta_1}] = \sigma_{\theta_1} \frac{\psi'(\theta_1)}{\psi'(\theta_2)}.$$

Consider the ascension time (or explosion time):

$$A = \inf\{\theta \in \Theta^\psi,\ \sigma_\theta < \infty\}, \tag{36}$$

where we use the convention $\inf \emptyset = \theta_\infty$. The following Theorem gives the distribution of the ascension time A and the distribution of the tree at this random time. Recall that $\bar{\theta} = \psi^{-1}(\psi(\theta))$ is defined at the end of Section 2.14.

Theorem 2.29 ([3]). *Assume that ψ is critical and satisfies Assumptions 1 and 2.*

1. *For all $\theta \in \Theta^\psi$, we have $\mathbb{N}^\psi[A > \theta] = \bar{\theta} - \theta$.*
2. *If $\theta_\infty < \theta < 0$, under $\mathbb{N}^\psi$, we have, for any non-negative measurable functional F,*

$$\mathbb{N}^\psi[F(\mathcal{T}_{A+\theta'}, \theta' \geq 0) | A = \theta] = \psi'(\bar{\theta}) \mathbb{N}^\psi[F(\mathcal{T}_{\theta'}, \theta' \geq 0) \sigma_0 \,\mathrm{e}^{-\psi(\theta)\sigma_0}].$$

3. *For all $\theta \in \Theta^\psi$, we have $\mathbb{N}^\psi[\sigma_A < +\infty | A = \theta] = 1$.*

In other words, at the ascension time, the tree can be seen as a size-biased critical Lévy tree. A precise description of $\mathcal{T}_A$ is given in [3]. Notice that in the setting of [3], there is no need of Assumption 2.

3. The growing tree-valued process

3.1. **Special Markov Property of pruning.** In [5], the authors prove a formula describing the structure of a Lévy tree, conditionally on the θ-pruned tree obtained from it in the (sub)critical case. We will give a general version of this result. From the measure of marks, $\mathcal{M}$ in (34), we define a measure of increasing marks by:

$$\mathcal{M}^{\uparrow}(dx, d\theta') = \sum_{i \in I^{\uparrow}} \delta_{(x_i, \theta_i)}(dx, d\theta'), \tag{37}$$

with

$$I^{\uparrow} = \left\{ i \in I^{\text{ske}} \cup I^{\text{nod}}; \mathcal{M}(\llbracket \emptyset, x_i \rrbracket \times [0, \theta_i]) = 1 \right\}.$$

The atoms (x_i, θ_i) for $i \in I^{\uparrow}$ correspond to marks such that there are no marks of $\mathcal{M}$ on $\llbracket \emptyset, x_i \rrbracket$ with a θ-component smaller than θ_i. In the case of multiple θ_j for a given node $x_i \in \text{Br}_\infty(\mathcal{T})$, we only keep the smallest one. In the case $\Pi = 0$, the measure $\mathcal{M}^{\uparrow}$ describes the jumps of a record process on the tree, see [1] for further work in this direction. The θ-pruned tree can alternatively be defined using $\mathcal{M}^{\uparrow}$ instead of $\mathcal{M}$ as for $\theta \geq 0$:

$$\Lambda_\theta(\mathcal{T}, \mathcal{M}) = \{x \in \mathcal{T},\ \mathcal{M}^{\uparrow}(\llbracket \emptyset, x \llbracket \times [0, \theta]) = 0\}.$$

We set:

$$I_\theta^{\uparrow} = \left\{ i \in I^{\uparrow}, x_i \in \text{Lf}(\Lambda_\theta(\mathcal{T}, \mathcal{M})) \right\} = \left\{ i \in I^{\uparrow}, \theta_i < \theta \quad \text{and} \quad \mathcal{M}^{\uparrow}(\llbracket \emptyset, x_i \llbracket \times [0, \theta]) = 0 \right\}.$$

and for $i \in I_\theta^{\uparrow}$:

$$\mathcal{T}^i = \mathcal{T} \backslash \mathcal{T}^{\emptyset, x_i} = \{x \in \mathcal{T},\ x_i \in \llbracket \emptyset, x \rrbracket\},$$

where $\mathcal{T}^{y,x}$ is the connected component of $\mathcal{T} \backslash \{x\}$ containing y. For $i \in I_\theta^{\uparrow}$, $\mathcal{T}^i$ is a real tree, and we will consider x_i as its root. The metric and mass measure on $\mathcal{T}^i$ are the restriction of the metric and mass measure of $\mathcal{T}$ on $\mathcal{T}^i$. By construction, we have:

$$\mathcal{T} = \Lambda_\theta(\mathcal{T}, \mathcal{M}) \circledast_{i \in I_\theta^{\uparrow}} (\mathcal{T}^i, x_i). \tag{38}$$

Now we can state the general special Markov property.

Theorem 3.1 (Special Markov Property)**.** *Let ψ be a branching mechanism satisfying Assumptions 1 and 2. Let $\theta > 0$. Conditionally on $\Lambda_\theta(\mathcal{T}, \mathcal{M})$, the point measure:*

$$\mathcal{M}_\theta^{\uparrow}(dx, d\mathcal{T}', d\theta') = \sum_{i \in I_\theta^{\uparrow}} \delta_{(x_i, \mathcal{T}^i, \theta_i)}(dx, d\mathcal{T}', d\theta')$$

under $\mathbb{P}_{r_0}^{\psi}$ (or under $\mathbb{N}^{\psi}$) is a Poisson point measure on $\Lambda_\theta(\mathcal{T}, \mathcal{M}) \times \mathbb{T} \times (0, \theta]$ with intensity:

$$\mathbf{m}^{\Lambda_\theta(\mathcal{T}, \mathcal{M})}(dx) \Big(2\beta \mathbb{N}^{\psi}[d\mathcal{T}'] + \int_{(0,+\infty)} \Pi(dr)\, r\, \mathrm{e}^{-\theta' r}\, \mathbb{P}_r^{\psi}(d\mathcal{T}') \Big)\, \mathbf{1}_{(0,\theta]}(\theta')\, d\theta'. \tag{39}$$

Proof. It is not difficult to adapt the proof of the special Markov property in [5] to get Theorem 3.1 in the (sub)critical case by taking into account the pruning times θ_i and the w-tree setting; and we omit this proof. We prove how to extend the result to the super-critical Lévy trees using the Girsanov transform of Definition 2.22.

Assume that ψ is super-critical. For $a > 0$, we shall write $\Lambda_{\theta,a}(\mathcal{T}, \mathcal{M}) = \pi_a(\Lambda_\theta(\mathcal{T}, \mathcal{M}))$ for short. According to (38) and the definition of super-critical Lévy trees, we have that for any $a > 0$, the truncated tree $\pi_a(\mathcal{T})$ can be written as:

$$\pi_a(\mathcal{T}) = \Lambda_{\theta,a}(\mathcal{T}, \mathcal{M}) \circledast_{\substack{i \in I_\theta^{\uparrow}, \\ H_{x_i} \leq a}} (\pi_{a - H_{x_i}}(\mathcal{T}^i), x_i)$$

and we have to prove that $\sum_{i\in I_\theta^{\uparrow,1}} \delta_{(x_i,\mathcal{T}^i,\theta_i)}(dx,d\mathcal{T}',d\theta')$ is conditionally on $\Lambda_\theta(\mathcal{T},\mathcal{M})$ a Poisson point measure with intensity (39). Since a is arbitrary, it is enough to prove that the point measure $\mathcal{M}_a$, defined by

$$\mathcal{M}_a(dx,d\mathcal{T}',d\theta') = \sum_{i\in I_\theta^\uparrow} \mathbf{1}_{\{H_{x_i}\leq a\}}\, \delta_{(x_i,\pi_{a-H_{x_i}}(\mathcal{T}^i),\theta_i)}(dx,d\mathcal{T}',d\theta'),$$

is conditionally on $\Lambda_{\theta,a}(\mathcal{T},\mathcal{M})$ a Poisson point measure with intensity :

$$\mathbf{1}_{[0,a]}(H_x)\, \mathbf{m}^{\Lambda_\theta(\mathcal{T},\mathcal{M})}(dx)\, \mathbf{1}_{(0,\theta]}(\theta')\, d\theta' \Big(2\beta\mathbb{N}^\psi(\pi^{-1}_{a-H_x}(d\mathcal{T}')) + \int_{(0,+\infty)} \Pi(dr)\, r\,\mathrm{e}^{-\theta' r}\, \mathbb{P}^\psi_r(\pi^{-1}_{a-H_x}(d\mathcal{T}'))\Big). \tag{40}$$

Let θ^* be the unique real number such that $\psi'_{\theta^*}(0)=0$, that is, such that ψ_{θ^*} is critical. Let Φ be a non-negative, measurable functional on $\Lambda_{\theta,a}(\mathcal{T},\mathcal{M})\times\mathbb{T}\times(0,\theta]$ and let F be a non-negative measurable functional on $\mathbb{T}$. Let

$$B=\mathbb{N}^\psi[F(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))\exp(-\langle\mathcal{M}_a,\Phi\rangle)].$$

Thanks to Girsanov formula (25) and the special Markov property for critical branching mechanisms, we get:

$$\begin{aligned}
B&=\mathbb{N}^{\psi_{\theta^*}}\left[F(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))\exp(-\langle\mathcal{M}_a,\Phi\rangle)\exp\left(\theta^*\mathcal{Z}_a(\mathcal{T})+\psi(\theta^*)\int_0^a \mathcal{Z}_h(\mathcal{T})dh\right)\right]\\
&=\mathbb{N}^{\psi_{\theta^*}}\Big[F(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))\exp\Big(\theta^*\mathcal{Z}_a(\Lambda_\theta(\mathcal{T},\mathcal{M}))+\psi(\theta^*)\int_0^a \mathcal{Z}_h(\Lambda_\theta(\mathcal{T},\mathcal{M}))dh\Big)\\
&\qquad\qquad \exp\Big(-\int \mathbf{m}^{\Lambda_{\theta,a}(\mathcal{T},\mathcal{M})}(dx)G(H_x,x,\theta)\Big)\Big],
\end{aligned}$$

with $\mathbf{m}^{\Lambda_{\theta,a}(\mathcal{T},\mathcal{M})}(dx)=\mathbf{1}_{[0,a]}(H_x)\,\mathbf{m}^{\Lambda_\theta(\mathcal{T},\mathcal{M})}(dx)$ and $G(h,x,\theta)$ equal to:

$$\begin{aligned}
&\int_0^\theta d\theta'\,\Big\{2\beta\mathbb{N}^{\psi_{\theta^*}}\left[1-\exp\left(-\Phi(x,\pi_{a-h}(\mathcal{T}),\theta')+\theta^*\mathcal{Z}_{a-h}(\mathcal{T})+\psi(\theta^*)\int_0^{a-h}\mathcal{Z}_t(\mathcal{T})dt\right)\right]\\
&\quad+\int_{(0,+\infty)}\Pi_{\theta^*}(dr)r\,\mathrm{e}^{-\theta' r}\,\mathbb{E}_r^{\psi_{\theta^*}}\left[1-\exp(-\Phi(x,\pi_{a-h}(\mathcal{T}),\theta')+\theta^*\mathcal{Z}_{a-h}(\mathcal{T})+\psi(\theta^*)\int_0^{a-h}\mathcal{Z}_t(\mathcal{T})dt)\right]\Big\}.
\end{aligned}$$

By using the Poisson decomposition of $\mathbb{P}_r^{\psi_{\theta^*}}$ (Proposition 2.15), we see that $G(h,x,\theta)$ can be written as:

$$G(h,x,\theta)=\int_0^\theta d\theta'\Big\{2\beta g(h,x,\theta')+\int_{(0,\infty)}\Pi_{\theta^*}(dr)\; r\,\mathrm{e}^{-\theta' r}\,(1-\exp(-rg(h,x,\theta')))\Big\},$$

with

$$g(h,x,\theta')=\mathbb{N}^{\psi_{\theta^*}}\left[1-\exp\left(-\Phi(x,\pi_{a-h}(\mathcal{T}),\theta')+\theta^*\mathcal{Z}_{a-h}(\mathcal{T})+\psi(\theta^*)\int_0^{a-h}\mathcal{Z}_t(\mathcal{T})dt\right)\right].$$

Thanks to the Girsanov formula and (33), we get:

$$\begin{aligned}
g(h,x,\theta')&=\mathbb{N}^{\psi_{\theta^*}}\Big[(1-\exp(-\Phi(x,\pi_{a-h}(\mathcal{T}),\theta')))\exp\Big(\theta^*\mathcal{Z}_{a-h}(\mathcal{T})+\psi(\theta^*)\int_0^{a-h}\mathcal{Z}_t(\mathcal{T})dt\Big)\Big]\\
&\qquad+\mathbb{N}^{\psi_{\theta^*}}\Big[1-\exp\Big(\theta^*\mathcal{Z}_{a-h}(\mathcal{T})+\psi(\theta^*)\int_0^{a-h}\mathcal{Z}_t(\mathcal{T}))dt\Big)\Big]\\
&=\mathbb{N}^\psi\Big[1-\exp(-\Phi(x,\pi_{a-h}(\mathcal{T}),\theta'))\Big]-\theta^*.
\end{aligned}$$

With $\tilde{g}(h,x,\theta') = \mathbb{N}^\psi\Big[1-\exp(-\Phi(x,\pi_{a-h}(\mathcal{T}),\theta'))\Big]$ and thanks to (11), we get:

$$G(h,x,\theta) = \int_0^\theta d\theta' \Big\{ 2\beta\tilde{g}(h,x,\theta') + \int_{(0,\infty)} \Pi(dr)\; r\,\mathrm{e}^{-\theta' r}\,(1-\exp(-r\tilde{g}(h,x,\theta')))\Big\} \\ + \psi(\theta^*) - \psi_\theta(\theta^*).$$

Notice that from the definition of G we have g replaced by $\tilde{g}$, Π_{θ^*} replaced by Π and the additional term $\psi(\theta^*) - \psi_\theta(\theta^*)$. As $\int \mathbf{m}^{\Lambda_{\theta,a}(\mathcal{T},\mathcal{M})}(dx) = \int_0^a \mathcal{Z}_h(\Lambda_\theta(\mathcal{T}))dh$, we get:

$$B = \mathbb{N}^{\psi_{\theta^*}}\Big[F(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))R(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M})) \\ \exp\Big(\theta^*\mathcal{Z}_a(\Lambda_\theta(\mathcal{T},\mathcal{M})) + \psi_\theta(\theta^*)\int_0^a \mathcal{Z}_h(\Lambda_\theta(\mathcal{T},\mathcal{M}))dh\Big)\Big], \tag{41}$$

with

$$R(\mathcal{T}) = \exp\Big(-\int \mathbf{m}^{\mathcal{T}}(dx)\int_0^\theta d\theta'\Big[2\beta\tilde{g}(H_x,x,\theta') + \\ \int_{(0,\infty)} \Pi(dr)\; r\,\mathrm{e}^{-\theta' r}\,(1-\exp(-r\tilde{g}(H_x,x,\theta')))\Big]\Big). \tag{42}$$

Taking $\Phi = 0$ (and thus $R = 1$) in (41) yields:

$$\mathbb{N}^\psi[F(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))] \\ = \mathbb{N}^{\psi_{\theta^*}}\Big[F(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))\exp\Big(\theta^*\mathcal{Z}_a(\Lambda_\theta(\mathcal{T},\mathcal{M})) + \psi_\theta(\theta^*)\int_0^a \mathcal{Z}_h(\Lambda_\theta(\mathcal{T},\mathcal{M}))dh\Big)\Big]. \tag{43}$$

Using (43) with F replaced by FR gives:

$$\mathbb{N}^\psi\Big[\exp(-\langle\mathcal{M}_a,\Phi\rangle)F(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))\Big] = B = \mathbb{N}^\psi\left[F(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))R(\Lambda_{\theta,a}(\mathcal{T},\mathcal{M}))\right].$$

This implies that $\mathcal{M}_a$ is, conditionally on $\Lambda_{\theta,a}(\mathcal{T},\mathcal{M})$, a Poisson point measure with intensity (40). This ends the proof. □

3.2. **An explicit construction of the growing process.** In this section, we will construct the growth process using a family of Poisson point measures. Let ψ be a branching mechanism satisfying Assumptions 1 and 2. Let $\theta \in \Theta^\psi$. According to (23) and (11), we have:

$$\mathbf{N}^{\psi_\theta}[\mathcal{T}\in\bullet] = 2\beta\mathbb{N}^{\psi_\theta}[\mathcal{T}\in\bullet] + \int_{(0,+\infty)} \Pi(dr) r\,\mathrm{e}^{-\theta r}\,\mathbb{P}_r^{\psi_\theta}(\mathcal{T}\in\bullet). \tag{44}$$

Let $\mathcal{T}^{(0)} \in \mathbb{T}$ with root $\emptyset$. For $q \in \Theta^\psi$ and $q \leq \theta$, we set:

$$\mathfrak{T}_q^{(0)} = \mathcal{T}^{(0)} \quad \text{and} \quad \mathbf{m}_q^{(0)} = \mathbf{m}^{\mathcal{T}(0)}.$$

We define the w-trees grafted on $\mathcal{T}^{(0)}$ by recursion on their generation. We suppose that all the random point measures used for the next construction are defined on $\mathbb{T}$ under a probability measure $Q^{\mathcal{T}^{(0)}}(d\omega)$.

Suppose that we have constructed the family of trees and mass measures $((\mathfrak{T}_q^{(k)},\mathbf{m}_q^{(n)}), 0 \leq k \leq n, q \in \Theta^\psi \cap (-\infty,\theta))$. We write

$$\mathfrak{T}^{(n)} = \bigsqcup_{q\in\Theta^\psi,\ q\leq\theta} \mathfrak{T}_q^{(n)}.$$

We define the $(n+1)$-th generation as follows. Conditionally on all trees from generations smaller than n, $(\mathfrak{T}_q^{(k)},\ 0\leq k\leq n,\ q\in\Theta^\psi\cap(-\infty,\theta))$, let

$$\mathcal{N}_\theta^{n+1}(dx,d\mathcal{T},dq)=\sum_{j\in J^{(n+1)}}\delta_{(x_j,\mathcal{T}^j,\theta_j)}(dx,d\mathcal{T},dq)$$

be a Poisson point measure on $\mathfrak{T}^{(n)}\times\mathbb{T}\times\Theta^\psi$ with intensity:

$$\mu_\theta^{n+1}(dx,d\mathcal{T},dq)=\mathbf{m}_q^{(n)}(dx)\mathbf{N}^{\psi_q}[d\mathcal{T}]\ \mathbf{1}_{\{q\leq\theta\}}\ dq.$$

For $q\in\Theta^\psi$ and $q\leq\theta$, we set

$$J_q^{(n+1)}=\{j\in J^{(n+1)},\ q<\theta_j\}$$

and we define the tree $\mathfrak{T}_q^{(n+1)}$ and the mass measure $\mathbf{m}_q^{(n+1)}$ by:

$$\mathfrak{T}_q^{(n+1)}=\mathfrak{T}_q^{(n)}\circledast_{j\in J_q^{(n+1)}}(\mathcal{T}^j,x_j)\quad\text{and}\quad\mathbf{m}_q^{(n+1)}=\sum_{j\in J_q^{(n+1)}}\mathbf{m}^{\mathcal{T}^j}(dx).$$

Notice that by construction, $(\mathfrak{T}_q^{(n)},n\in\mathbb{N})$ is a non-decreasing sequence of trees. We set $\mathfrak{T}_q$ the completion of $\cup_{n\in\mathbb{N}}\mathfrak{T}_q^{(n)}$, which is a real tree with root $\emptyset$ and obvious metric $d^{\mathfrak{T}_q}$, and we define a mass measure on $\mathfrak{T}_q$ by $\mathbf{m}^{\mathfrak{T}_q}=\sum_{n\in\mathbb{N}}\mathbf{m}_q^{(n)}$.

For $q\in\Theta^\psi$ and $q<\theta$, we consider $\mathcal{F}_q$ the σ-field generated by $\mathfrak{T}^{(0)}$ and the sequence of random point measures $(\mathbf{1}_{\{q'\in[q,\theta]\}}\mathcal{N}_\theta^{(n)}(dx,d\mathcal{T},dq'),n\in\mathbb{N})$. We set $\mathcal{N}_\theta=\sum_{n\in\mathbb{N}}\mathcal{N}_\theta^n$. The backward random point process $q\mapsto\mathbf{1}_{\{q\leq q'\}}\mathcal{N}_\theta(dx,d\mathcal{T},dq')$ is by construction adapted to the backward filtration $(\mathcal{F}_q,q\in\Theta^\psi\cap(-\infty,\theta])$.

The proof of the following result is postponed to Section 3.3.

Theorem 3.2. *Under $Q^{\psi_\theta}:=\mathbb{N}^{\psi_\theta}[d\mathcal{T}^{(0)}]Q^{\mathcal{T}^{(0)}}(d\omega)$, the process*

$$((\mathfrak{T}_q,d^{\mathfrak{T}_q},\emptyset,\mathbf{m}^{\bar{\mathfrak{T}}_q}),q\in\Theta^\psi\cap(-\infty,\theta])$$

is a $\mathbb{T}$-valued backward Markov process with respect to the backward filtration $\mathcal{F}^\theta=(\mathcal{F}_q,q\in\Theta^\psi\cap(-\infty,\theta])$. It is distributed as $((\mathcal{T}_q,\mathbf{m}^{\mathcal{T}_q}),q\in\Theta^\psi\cap(-\infty,\theta])$ under $\mathbb{N}^\psi$.

Notice the Theorem in particular entails that $(\mathfrak{T}_q,d^{\mathfrak{T}_q},\emptyset,\mathbf{m}^{\bar{\mathfrak{T}}_q})$ is a w-tree for all q.

We shall use the following Lemma.

Lemma 3.3. *Let K be a measurable non-negative process (as a function of q) defined on $\mathbb{R}_+\times\mathbb{T}\times\mathbb{T}$ which is predictable with respect to the backward filtration $\mathcal{F}^\theta$. We have:*

$$Q^{\psi_\theta}\left[\int\mathcal{N}_\theta(dx,d\mathcal{T},dq)\ K(q,\mathfrak{T}_q,\mathfrak{T}_{q-})\right]=Q^{\psi_\theta}\left[\int K\Big(q,\mathfrak{T}_q,\mathfrak{T}_q\circledast(\mathcal{T},x)\Big)\ \mu_\theta(dx,d\mathcal{T},dq)\right],$$

where $\mu_\theta(dx,d\mathcal{T},dq)=\sum_{n\in\mathbb{N}^*}\mu^n(dx,dT,dq)=\mathbf{m}^{\mathfrak{T}_q}(dx)\mathbf{N}^{\psi_q}[d\mathcal{T}]\ \mathbf{1}_{\{q\in\Theta^\psi,q\leq\theta\}}\ dq$.

This means that the predictable compensator of $\mathcal{N}_\theta$ is given by:

$$\mu_\theta(dx,d\mathcal{T},dq)=\mathbf{m}^{\mathfrak{T}_q}(dx)\mathbf{N}^{\psi_q}[d\mathcal{T}]\ \mathbf{1}_{\{q\in\Theta^\psi,q\leq\theta\}}\ dq.$$

Notice this construction does not fit in the usual framework of random point measures as the support at time q of the predictable compensator is the (predictable backward in time) random set $\mathfrak{T}_q\times\mathbb{T}\times\Theta^\psi$.

Proof. Based on the recursive construction, we have:

$$Q^{\psi_\theta}\left[\int \mathcal{N}_\theta(dx, d\mathcal{T}, dq)\ K(q, \mathfrak{T}_q, \mathfrak{T}_{q-})\right]$$
$$= \sum_{n=0}^{+\infty} Q^{\psi_\theta}\left[Q^{\psi_\theta}\left[\int \mathcal{N}_\theta^n(dx, d\mathcal{T}, dq)\ K(q, \mathfrak{T}_q, \mathfrak{T}_q \circledast (\mathcal{T}, x)) \,\Big|\, (\mathfrak{T}_s^{(k)},\ k \le n,\ s \le \theta)\right]\right].$$

Now, by construction, we have that:

$$\mathfrak{T}_q = \mathfrak{T}_q^{(n)} \circledast_{j \in J_q^{(n)}} (\tilde{\mathcal{T}}_j, x_j)$$

for $\tilde{\mathcal{T}}_j = \mathfrak{T}_q \backslash \mathfrak{T}_q^{(x_j, \emptyset)}$ which is a measurable function of $\mathbf{1}_{\{q' > q\}} \mathcal{N}_\theta^n(dx, d\mathcal{T}, dq')$ and of the point measures $\mathbf{1}_{\{q'>q\}} \mathcal{N}_\theta^\ell(dx, d\mathcal{T}, dq')$ for $\ell \ge n+1$. Therefore, applying the Palm formula with the function

$$F_n\Big(q, \mathcal{T}, x, \sum_{j \in J^{(n)}, q_j > q} \delta_{(x_j, \mathcal{T}^j, \theta_j)}\Big) = Q^{\psi_\theta}\Big[K\Big(q, \mathfrak{T}_q^{(n)} \circledast_{j \in J_q^{(n)}} (\tilde{\mathcal{T}}_j, x_j),$$
$$\mathfrak{T}_q^{(n)} \circledast_{j \in J_q^{(n)}} (\tilde{\mathcal{T}}_j, x_j) \circledast (\mathcal{T}, x)\Big) \,\Big|\, (\mathfrak{T}_s^{(k)},\ k \le n,\ s \le \theta), \mathcal{N}_\theta^n\Big],$$

we get:

$$Q^{\psi_\theta}\left[\int \mathcal{N}_\theta(dx, d\mathcal{T}, dq)\ K(q, \mathfrak{T}_q, \mathfrak{T}_{q-})\right]$$
$$= \sum_{n=0}^{+\infty} Q^{\psi_\theta}\Big[Q^{\psi_\theta}\Big[\int \mathcal{N}_\theta^n(dx, d\mathcal{T}, dq)$$
$$F_n\Big(q, \mathcal{T}, x, \sum_{j \in J^{(n)}, q_j > q} \delta_{(x_j, \mathcal{T}^j, \theta_j)}\Big) \,\Big|\, (\mathfrak{T}_s^{(k)},\ k \le n,\ s \le \theta)\Big]\Big]$$
$$= \sum_{n=0}^{+\infty} Q^{\psi_\theta}\Big[Q^{\psi_\theta}\Big[\int \mu_\theta^n(dx, d\mathcal{T}, dq)$$
$$F_n\Big(q, \mathcal{T}, x, \sum_{j \in J^{(n)}, q_j > q} \delta_{(x_j, \mathcal{T}^j, \theta_j)}\Big) \,\Big|\, (\mathfrak{T}_s^{(k)},\ k \le n,\ s \le \theta)\Big]\Big]$$
$$= \sum_{n=0}^{+\infty} Q^{\psi_\theta}\left[Q^{\psi_\theta}\left[\int \mu_\theta^n(dx, d\mathcal{T}, dq)\ K\Big(q, \mathfrak{T}_q^{(n)} \circledast_{j \in J_q^{(n)}} (\tilde{\mathcal{T}}_j, x_j),\right.\right.$$
$$\left.\left.\mathfrak{T}_q^{(n)} \circledast_{j \in J_q^{(n)}} (\tilde{\mathcal{T}}_j, x_j) \circledast (\mathcal{T}, x)\Big) \,\Big|\, (\mathfrak{T}_s^{(k)},\ k \le n,\ s \le \theta)\right]\right]$$
$$= \sum_{n=0}^{+\infty} Q^{\psi_\theta}\left[\int \mu_\theta^n(dx, d\mathcal{T}, dq)\ K\Big(q, \mathfrak{T}_q, \mathfrak{T}_q \circledast (\mathcal{T}, x)\Big)\right]$$
$$= Q^{\psi_\theta}\left[\int K\Big(q, \mathfrak{T}_q, \mathfrak{T}_q \circledast (T, x)\Big)\ \mu_\theta(dx, d\mathcal{T}, dq)\right].$$

□

It can be noticed that the map $q \mapsto \mathfrak{T}_q$ is non-decreasing càdlàg (backwards in time) and that we have, for $j \in \cup_{n \in \mathbb{N}} J^{(n)}$, $x_j \in \mathfrak{T}_{\theta_j}$: $\mathfrak{T}_{\theta_j -} = \mathfrak{T}_{\theta_j} \circledast (\mathcal{T}^j, x_j)$. In particular, we can recover the random measure $\mathcal{N}_\theta$ from the jumps of the process $(\mathfrak{T}_q, q \in \Theta^\psi \cap (-\infty, \theta])$. This and the natural compatibility relation of $\mathcal{N}_\theta$ with respect to θ gives the next Corollary.

Corollary 3.4. *Let $(\mathcal{T}_\theta, \theta \in \Theta^\psi)$ be defined under $\mathbb{N}^\psi$. Let*

$$\mathcal{N} = \sum_{j \in J} \delta_{(x_j, \mathcal{T}^j, \theta_j)}$$

be the random point measure defined as follows:

- *The set $\{\theta_j; j \in J\}$ is the set of jumping times of the process $(\mathcal{T}_\theta, \theta \in \Theta^\psi)$: for $j \in J$, $\mathcal{T}_{\theta_j -} \neq \mathcal{T}_{\theta_j}$.*
- *The real tree $\mathcal{T}^j$ is the closure of $\mathcal{T}_{\theta_j -} \setminus \mathcal{T}_{\theta_j}$.*
- *The point x_j is the root of $\mathcal{T}^j$ (that is x_j is the only element $y \in \mathcal{T}_{\theta_j -}$ such that $x \in \mathcal{T}^j$ implies $[\![y, x]\!] \subset \mathcal{T}^j$).*

Then the backward point process $\theta \mapsto \mathbf{1}_{\{\theta \leq q'\}} \mathcal{N}_\theta(dx, d\mathcal{T}, dq')$ defined on Θ^ψ has predictable compensator:

$$\mu(dx, d\mathcal{T}, dq) = \mathbf{m}^{\mathcal{T}_q}(dx) \mathbf{N}^{\psi_q}[d\mathcal{T}] \, \mathbf{1}_{\{q \in \Theta^\psi\}} \, dq,$$

with respect to the backward left-continuous filtration $\mathcal{F} = (\mathcal{F}_\theta, \theta \in \Theta^\psi)$ defined by:

$$\mathcal{F}_\theta = \sigma((x_j, \mathcal{T}^j, \theta_j); \theta \leq \theta_j) = \sigma(\mathcal{T}_{q-}; \theta \leq q).$$

More precisely, for any non-negative predictable process K with respect to the backward filtration $\mathcal{F}$, we have:

$$\text{(45)} \quad \mathbb{N}^\psi \left[\int \mathcal{N}(dx, d\mathcal{T}, dq) \, K\Big(q, \mathcal{T}_q, \mathcal{T}_{q-}\Big) \right] = \mathbb{N}^\psi \left[\int \mu_\theta(dx, dT, dq) \, K\Big(q, \mathcal{T}_q, \mathcal{T}_q \circledast (T, x)\Big) \right].$$

Remark 3.5. Notice that Assumption 2 is assumed only for technical measurability condition, see Remark 2.12. We conjecture that this results holds also if Assumption 2 is not in force.

As a consequence, thanks to property 3 of Theorem 2.29, we get, with the convention $\sup \emptyset = \theta_\infty$, that:

$$A = \sup\{\theta_j, j \in J \text{ and } \sigma^j = +\infty\} \quad \text{with} \quad \sigma_j = \mathbf{m}^{\mathcal{T}^j}(\mathcal{T}^j).$$

3.3. **Proof of Theorem 3.2.** By construction, it is clear that the process $(\mathfrak{T}_q, q \in \Theta^\psi \cap (-\infty, \theta])$ is a backward Markov process with respect to the backward filtration $(\mathcal{F}_q, q \in \Theta^\psi \cap (-\infty, \theta])$. By construction this process is càglàd in backward time. Since the process $(\mathcal{T}_q, q \in \Theta^\psi)$ is a forward càdlàg Markov process, it is enough to check that for $\theta_0 \in \Theta^\psi$, such that $\theta_0 < \theta$, the two dimensional marginals $(\mathfrak{T}_{\theta_0}, \mathfrak{T}_\theta)$ and $(\mathcal{T}_{\theta_0}, \mathcal{T}_\theta)$ have the same distribution.

Replacing ψ by ψ_{θ_0}, we can assume that $\theta_0 = 0$ and $0 < \theta$. We shall decompose the big tree $\mathcal{T}_0$ conditionally on the small tree $\mathcal{T}_\theta$ by iteration. This decomposition is similar to the one which appears in [2] or [24] for the fragmentation of the (sub)critical Lévy tree, but roughly speaking the fragmentation is here frozen but for the fragment containing the root.

We set $\mathcal{T}^{(0)} = \mathcal{T}_\theta$ and $\tilde{\mathbf{m}}^{(0)} = \mathbf{m}^{\mathcal{T}_\theta}$, so that $(\mathfrak{T}^{(0)}, \mathbf{m}^{(0)})$ and $(\mathcal{T}^{(0)}, \tilde{\mathbf{m}}^{(0)})$ have the same distribution. Recall notation $\mathcal{M}^\uparrow$ from (37) as well as (38): $\mathcal{T}_0 = \mathcal{T}^{(0)} \circledast_{i \in I_\theta^{\uparrow,1}} (\mathcal{T}^i, x_i)$, where we write $I_\theta^{\uparrow,1} = I_\theta^\uparrow$ and where $\mathcal{P}^1 = \sum_{i \in I_\theta^{\uparrow,1}} \delta_{(x_i, \mathcal{T}^i, \theta_i)}$ is, conditionally on $\mathcal{T}^{(0)}$, a Poisson point measure with intensity:

$$\nu^1(dx, d\mathcal{T}', dq) = \tilde{\mathbf{m}}^{(0)}(dx) \Big(2\beta \mathbb{N}^\psi[d\mathcal{T}'] + \int_{(0,+\infty)} \Pi(dr) \, r \, \mathrm{e}^{-qr} \, \mathbb{P}_r^\psi(d\mathcal{T}') \Big) \, \mathbf{1}_{(0,\theta]}(q) \, dq.$$

For $i \in I_\theta^{\uparrow,1}$, we define the sub-tree of $\mathcal{T}^i$:

$$\tilde{\mathcal{T}}^i = \{x \in \mathcal{T}^i; \mathcal{M}^\uparrow(]\!]x_i, x[\![\times [0, \theta_i]) = 0\}.$$

Since $\mathcal{T}^i$ is distributed according to $\mathbb{N}^\psi$ (or to $\mathbb{P}^\psi_{r_i}$ for some $r_i > 0$), using the property of Poisson point measures, we have that conditionally on $\mathcal{T}^0$ and θ_i, the tree $\tilde{\mathcal{T}}^i$ is distributed as $\Lambda_{\theta_i}(\mathcal{T}, \mathcal{M})$ under $\mathbb{N}^\psi$ (or under $\mathbb{P}^\psi_{r_i}$) that is the distribution of $\tilde{\mathcal{T}}^i$ is $\mathbb{N}^{\psi_{\theta_i}}[d\mathcal{T}]$ (or $\mathbb{P}^{\psi_{\theta_i}}_{r_i}(d\mathcal{T})$), thanks to the special Markov property. Furthermore we have $\mathcal{T}^i = \tilde{\mathcal{T}}^i \circledast_{i' \in I^{\uparrow,2}_{\theta,i}} (\mathcal{T}^{i'}, x_{i'})$ where

$$\sum_{i' \in I^{\uparrow,2}_{\theta,i}} \delta_{(x_{i'}, \mathcal{T}^{i'}, \theta_{i'})}$$

is, conditionally on $\mathcal{T}^{(0)}$ and $\tilde{\mathcal{T}}^i$ a Poisson point measure on $\tilde{\mathcal{T}}^i \times \mathbb{T} \times (0, \theta]$ with intensity:

$$\mathbf{m}^{\tilde{\mathcal{T}}^i}(dx)\Big(2\beta \mathbb{N}^\psi(d\mathcal{T}') + \int_{(0,+\infty)} \Pi(dr)\, r\, \mathrm{e}^{-qr}\, \mathbb{P}^\psi_r(d\mathcal{T}')\Big)\, \mathbf{1}_{[0,\theta_i)}(q)\, dq.$$

Thus we deduce, using again the special Markov property, that:

$$\tilde{\mathcal{N}}^1_\theta(dx, d\mathcal{T}, dq) = \sum_{i \in I^{\uparrow,1}} \delta_{(x_i, \tilde{\mathcal{T}}^i, \theta_i)}(dx, d\mathcal{T}, dq)$$

is conditionally on $\mathcal{T}^0$ a Poisson point measure on $\mathcal{T}^{(0)} \times \mathbb{T} \times \Theta^\psi$ with intensity:

$$\tilde{\mu}^1(dx, d\mathcal{T}, dq) = \tilde{\mathbf{m}}^{(0)}_q(dx) \mathbf{N}^{\psi_q}[d\mathcal{T}]\, \mathbf{1}_{[0,\theta)}(q)\, dq,$$

with $\tilde{\mathbf{m}}^{(0)}_q(dx) = \tilde{\mathbf{m}}^{(0)}(dx)$. We set $\mathcal{T}^{(1)} = \mathcal{T}^{(0)} \circledast_{i \in I^{\uparrow,1}_\theta} (\tilde{\mathcal{T}}^i, x_i)$ for the first generation tree and for $q \in [0, \theta]$:

$$\tilde{\mathbf{m}}^{(1)}_q(dx) = \sum_{i \in I^{\uparrow,1}_\theta} \mathbf{m}^{\tilde{\mathcal{T}}^i}(dx) \mathbf{1}_{[0,\theta_i)}(q).$$

See Figure 2 for a simplified representation. We get that $(\mathfrak{T}^{(1)}_\theta, (\mathbf{m}^{(1)}_q, q \in [0, \theta]), \mathfrak{T}^{(0)}, \mathbf{m}^{\mathfrak{T}^{(0)}})$ and $(\mathcal{T}^{(1)}, (\tilde{\mathbf{m}}^{(1)}_q, q \in [0, \theta]), \mathcal{T}^{(0)}, \tilde{\mathbf{m}}^{(0)})$ have the same distribution.

FIGURE 2. The tree $\mathcal{T}_0$, $\mathcal{T}^{(0)}$, and a tree $\mathcal{T}^i$ and its sub-tree $\tilde{\mathcal{T}}^i$ belonging to the first generation tree $\mathcal{T}^{(1)} \setminus \mathcal{T}^{(0)}$.

Furthermore, by collecting all the trees grafted on $\mathcal{T}^{(1)}$, we get that $\mathcal{T} = \mathcal{T}^{(1)} \circledast_{i' \in I^{\uparrow,2}_\theta} (\mathcal{T}^{i'}, x_{i'})$ where $I^{\uparrow,2}_\theta = \cup_{i \in I^{\uparrow,1}_\theta} I^{\uparrow,2}_{\theta,i}$ and

$$\mathcal{P}^2 = \sum_{i' \in I^{\uparrow,2}_\theta} \delta_{(x_{i'}, \mathcal{T}^{i'}, \theta_{i'})}$$

is, conditionally on $(\mathcal{T}^{(1)}, (\tilde{\mathbf{m}}^{(1)}_q, q \in [0, \theta]), \mathcal{T}^{(0)}, \tilde{\mathbf{m}}^{(0)})$ a Poisson point measure on $\mathcal{T}^{(1)} \times \mathbb{T} \times (0, \theta]$ with intensity:

$$\nu^2(dx, d\mathcal{T}, dq) = \tilde{\mathbf{m}}^{(1)}_q(dx)\, \Big(2\beta \mathbb{N}^\psi(d\mathcal{T}') + \int_{(0,+\infty)} \Pi(dr)\, r\, \mathrm{e}^{-qr}\, \mathbb{P}^\psi_r(d\mathcal{T}')\Big)\, \mathbf{1}_{[0,\theta]}(q)\, dq.$$

Notice that:

$$\mathcal{T}^{(1)} = \{x \in \mathcal{T}_0; \mathcal{M}^\uparrow([\![\emptyset, x[\![\times [0, \theta]) \leq 1\} \quad \text{and} \quad \tilde{\mathbf{m}}^{(1)}_\theta(dx) + \tilde{\mathbf{m}}^{(0)}(dx) = \mathbf{1}_{\mathcal{T}^{(1)}}(x)\, \mathbf{m}^{\mathcal{T}_0}(dx). \tag{46}$$

Then we can iterate this construction, and by taking increasing limits we obtain that the pair $((\cup_{n\in\mathbb{N}}\mathfrak{T}_\theta^{(n)}, \sum_{n\in\mathbb{N}}\mathbf{m}_\theta^{(n)}), \mathfrak{T}_0)$ has the same distribution as $(\mathcal{T}', \mathcal{T}^{(0)})$, where:

$$\mathcal{T}' = \{x \in \mathcal{T}_0; \mathcal{M}^\uparrow([\![\emptyset, x[\![\times[0,\theta]) < +\infty\} \quad \text{and} \quad \tilde{\mathbf{m}}'(dx) = \mathbf{1}_{\mathcal{T}'}(x)\,\mathbf{m}^{\mathcal{T}_0}(dx).$$

To conclude, we need to check first that the completion of $\mathcal{T}'$ is $\mathcal{T}_0$, or as $\mathcal{T}_0$ is complete that the closure of $\mathcal{T}'$ as a subset of $\mathcal{T}_0$ is exactly $\mathcal{T}_0$ and then that $\mathbf{m}^{\mathcal{T}_0}(\mathcal{T}'^c) = 0$.

Notice that $\mathcal{M}^\uparrow$ has less marks than $\mathcal{M}$. Then Proposition 1.2 in [2] in the case when $\beta = 0$ or an elementary adaptation of it in the general framework of [24], gives there is no loss of mass in the fragmentation process. This implies that, if ψ is (sub)critical, then:

$$\mathbf{m}^{\mathcal{T}_0}(\{x \in \mathcal{T}_0; \mathcal{M}([\![\emptyset, x[\![\times[0,\theta]) = \infty\} = 0. \tag{47}$$

Then, if ψ is super-critical, by considering the restriction of $\mathcal{T}_0$ up to level a, $\pi_a(\mathcal{T}_0)$, and using a Girsanov transformation from Definition 2.22 with $\theta = \theta^*$ and (47), we deduce that (47) holds for $\pi_a(\mathcal{T}_0)$. Since a is arbitrary, we deduce by monotone convergence that (47) holds also in the super-critical case. Thus we have $\mathbf{m}^{\mathcal{T}_0}(\mathcal{T}'^c) = 0$. Since the closed support of $\mathbf{m}^{\mathcal{T}_0}$ is the set of leaves $\mathrm{Lf}(\mathcal{T}_0)$, we deduce that $\mathrm{Lf}(\mathcal{T}')$ is dense in $\mathrm{Lf}(\mathcal{T}_0)$ and, as $\mathcal{T}'$ and $\mathcal{T}_0$ have the same root, that $\mathrm{Sk}(\mathcal{T}') = \mathrm{Sk}(\mathcal{T}_0)$. This implies that the closure of $\mathcal{T}'$ is $\mathcal{T}_0$. This ends the proof.

4. Application to overshooting

We assume that ψ is critical, $\theta_\infty < 0$ and Assumptions 1 and 2 hold. We shall write u^θ (resp. b^θ) for the solution of (14) (resp. (15)) when ψ is replaced by ψ_θ, for $a \geq 0$, $h > 0$ and $t \in [0, h)$:

$$\int_{u^\theta(a,\lambda)}^{\lambda} \frac{dr}{\psi_\theta(r)} = a, \quad \text{and} \quad b_h^\theta(t) = b^\theta(h-t) \quad \text{with} \quad \int_{b^\theta(h)}^{\infty} \frac{dr}{\psi_\theta(r)} = h. \tag{48}$$

We have $u^\theta(a, b^\theta(h-a)) = b^\theta(h)$. Notice that $\partial_h b^\theta(h)/\psi_\theta(b^\theta(h)) = -1$ and that $\partial_\lambda u^\theta(a,\lambda) = \psi_\theta(u^\theta(a,\lambda))/\psi_\theta(\lambda)$ which implies that:

$$\partial_\lambda u^\theta(a, b^\theta(h-a)) = \frac{\psi_\theta(b^\theta(h))}{\psi_\theta(b^\theta(h-a))} = -\frac{\psi_\theta(b^\theta(h))}{\psi_\theta(b^\theta(h-a))^2}\partial_h b^\theta(h-a). \tag{49}$$

We set for $\theta \in \Theta^\psi$ and $\lambda \geq 0$:

$$\gamma_\theta(\lambda) = \psi_\theta'(\lambda) - \psi_\theta'(0) = \psi'(\lambda+\theta) - \psi'(\theta) = \partial_\theta \psi_\theta(\lambda). \tag{50}$$

Notice the function γ_θ is non-negative and non-decreasing.

Recall that $\bar{\theta} = \psi^{-1} \circ \psi(\theta)$. We deduce from (48) that for $\theta \in \Theta^\psi$, $\theta < 0$ and $h > 0$:

$$\bar{\theta} + b^{\bar{\theta}}(h) = \theta + b^\theta(h) \quad \text{and} \quad \psi_{\bar{\theta}}(b^{\bar{\theta}}(h)) = \psi_\theta(b^\theta(h)). \tag{51}$$

4.1. **Exit times .** Let $h > 0$. We are interested in the first time when the process of growing trees exceeds height h, in the following sense.

Definition 4.1. *The first exit time out of h is the (possibly infinite) number Θ_h defined by*

$$\Theta_h = \sup\{\theta \in \Theta^\psi,\ H_{max}(\mathcal{T}_\theta) > h\},$$

with the convention that $\sup\emptyset = \theta_\infty$.

The constraint not to be higher than h will be coded by the function $b^\theta(h)$ which is the probability (under $\mathbb{N}^\psi$) for the tree $\mathcal{T}^\theta$ of having maximal height larger than h. By definition of the function b, we have for $\theta \in \Theta^\psi$:

$$\mathbb{N}^\psi[\theta \leq \Theta_h] = \mathbb{N}^\psi\left[H_{max}(\mathcal{T}_\theta) \geq h\right] = b^\theta(h). \tag{52}$$

Proposition 4.2. *The function $\theta \mapsto b_h^\theta$ is of class $\mathcal{C}^1$ on $(\theta_\infty, +\infty)$. And, under $\mathbb{N}^\psi$, the distribution of Θ_h on $(\theta_\infty, +\infty)$ has density $\theta \mapsto -\partial_\theta b^\theta(h)$ with respect to the Lebesgue measure. We also have the following expression for the density of Θ_h on $(\theta_\infty, +\infty)$. Let $\theta_\infty < \theta$ and $h > 0$. Then:*

$$-\partial_\theta b^\theta(h) = \psi_\theta(b^\theta(h)) \int_0^h da \; \frac{\gamma_\theta(b^\theta(a))}{\psi_\theta(b^\theta(a))} = \int_0^h da \; \gamma_\theta(b^\theta(h-a)) \; \mathrm{e}^{-\psi'(\theta)a - \int_0^a dx \; \gamma_\theta(b^\theta(h-x))} .$$

Notice the distribution of Θ_h might have an atom at θ_∞.

Proof. Notice that for $\theta_\infty < \theta$, we have $\lim_{\lambda\to+\infty} \psi''(\lambda) = \beta$ and $\lim_{\lambda\to+\infty} \psi'(\lambda) = +\infty$. In particular $\psi'_\theta(\lambda)/\psi_\theta(\lambda)$ is bounded for λ large enough. This implies that $\int^{+\infty} dr \; \psi'_\theta(r)/\psi_\theta(r)^2$ is finite thanks to Assumption 2. We deduce that the function $\theta \mapsto b_h^\theta$ is of class $\mathcal{C}^1$ on $(\theta_\infty, +\infty)$ and, thanks to (52), that under $\mathbb{N}^\psi$, the distribution of Θ_h on $(\theta_\infty, +\infty)$ has density $\theta \mapsto -\partial_\theta b^\theta(h)$ with respect to the Lebesgue measure.

Taking the derivative with respect to θ in the last term of (48), using (50) and the change of variable $r = b^\theta(a)$ gives the first equality of the Lemma:

$$-\partial_\theta b^\theta(h) = \psi_\theta(b^\theta(h)) \int_{b^\theta(h)}^{+\infty} dr \frac{\gamma_\theta(r)}{\psi_\theta(r)^2} = \psi_\theta(b^\theta(h)) \int_0^h da \; \frac{\gamma_\theta(b^\theta(a))}{\psi_\theta(b^\theta(a))} \cdot \tag{53}$$

From (48) we get that $\partial_t b_h^\theta(t) = \psi_\theta(b_h^\theta(t))$. Hence, we have:

$$\int_0^t \psi'_\theta(b_h^\theta(r)) \; dr = \int_0^t \frac{\psi'_\theta(b_h^\theta(r))}{\psi_\theta(b_h^\theta(r))} \partial_t b_h^\theta(r) \; dr = \log\left(\frac{\psi_\theta(b_h^\theta(t))}{\psi_\theta(b_h^\theta(0))}\right).$$

This gives:

$$\int_0^t \gamma_\theta(b_h^\theta(r)) dr = \int_0^t \psi'_\theta(b_h^\theta(r)) \; dr - t\psi'(\theta) = \log\left(\frac{\psi_\theta(b_h^\theta(t))}{\psi_\theta(b_h^\theta(0))}\right) - t\psi'(\theta). \tag{54}$$

We deduce that:

$$\int_0^h da \; \gamma_\theta(b^\theta(h-a)) \; \mathrm{e}^{-\psi'(\theta)a - \int_0^a dx \; \gamma_\theta(b^\theta(h-x))} = \psi_\theta(b^\theta(h)) \int_0^h da \; \frac{\gamma_\theta(b^\theta(a))}{\psi_\theta(b^\theta(a))} \cdot$$

This proves the second equality of the Lemma. □

Since we will also be dealing with super-critical trees, there is always the positive probability that in the Poisson process of trees an infinite tree arises, which will be grafted onto the process, effectively making it infinite and thus outgrowing height h. In the next proposition, we will compute the conditional distribution of overshooting time Θ_h, given A. Note that we always have $A \leq \Theta_h$.

Proposition 4.3. *Under Assumptions 1 and 2, for $\theta_\infty < \theta_0 < \theta$ and $\theta_0 < 0$ (that is ψ_{θ_0} super-critical), we have, with $\hat{\theta} = \bar{\theta}_0 - \theta_0 + \theta$:*

$$\mathbb{N}^\psi[\Theta_h \geq \theta | A = \theta_0] = 1 - \psi'(\hat{\theta}) \psi_{\hat{\theta}}(b^{\hat{\theta}}(h)) \int_{b^{\hat{\theta}}(h)}^{+\infty} \frac{dr}{\psi_{\hat{\theta}}(r)^2},$$

$$\mathbb{N}^\psi[\Theta_h = A | A = \theta_0] = \psi'(\bar{\theta}_0) \psi_{\bar{\theta}_0}(b^{\bar{\theta}_0}(h)) \int_{b^{\bar{\theta}_0}(h)}^{+\infty} \frac{dr}{\psi_{\bar{\theta}_0}(r)^2} \cdot$$

Since $\psi_{\bar{\theta}_0}$ is sub-critical, we have $\psi'(\bar{\theta}_0) > 0$ and $\psi_{\bar{\theta}_0}(r) \sim r\psi'(\bar{\theta}_0)$ when r goes down to 0. Since $\lim_{h\to+\infty} b^{\bar{\theta}_0}(h) = 0$, we deduce that:

$$\lim_{h\to+\infty} \mathbb{N}^\psi[\Theta_h = A | A = \theta_0] = 1.$$

This has a straightforward explanation. If h is very large, with high probability the process up to A will not have crossed height h, so that the first jump to cross height h will correspond to the grafting time of the first infinite tree which happens at the ascension time A.

We also deduce from (51) that:

$$\mathbb{N}^\psi[\Theta_h = A|A = \theta_0] = \psi'(\bar{\theta}_0)\psi_{\theta_0}(b^{\theta_0}(h)) \int_{b^{\theta_0}(h)}^{+\infty} \frac{dr}{\psi_{\theta_0}(r)^2}. \tag{55}$$

Proof. We use the notation $\mathcal{Z}_h^\theta = \mathcal{Z}_h(\mathcal{T}^\theta)$ and $\mathcal{Z}_h = \mathcal{Z}_h(\mathcal{T}^0)$. We have:

$$\begin{aligned}
\mathbb{N}^\psi[\Theta_h \geq \theta|A = \theta_0] = \mathbb{N}^\psi[\mathcal{Z}_h^\theta > 0|A = \theta_0] &= \mathbb{N}^\psi[\mathcal{Z}_h^{A+(\theta-\theta_0)} > 0|A = \theta_0] \\
&= \psi'(\bar{\theta}_0)\mathbb{N}^\psi\left[\sigma_0 \mathbf{1}_{\{\mathcal{Z}_h^{(\theta-\theta_0)}>0\}}\, \mathrm{e}^{-\psi(\theta_0)\sigma_0}\right] \\
&= \psi'(\bar{\theta}_0)\mathbb{N}^{\psi_{\bar{\theta}_0}}\left[\sigma_0 \mathbf{1}_{\{\mathcal{Z}_h^{(\theta-\theta_0)}>0\}}\right] \\
&= \psi'(\bar{\theta}_0)\mathbb{N}^\psi\left[\sigma_{\bar{\theta}_0} \mathbf{1}_{\{\mathcal{Z}_h^{\bar{\theta}_0+(\theta-\theta_0)}>0\}}\right] \\
&= \psi'(\bar{\theta}_0)\mathbb{N}^\psi\left[\sigma_{\bar{\theta}_0} \mathbf{1}_{\{\mathcal{Z}_h^{\hat{\theta}}>0\}}\right],
\end{aligned}$$

where we used (2) of Theorem 2.29 for the third equality, Girsanov formula (30) for the fourth and the homogeneity property of Theorem 2.27 in the fifth. We now condition with respect to $\mathcal{T}^{\hat{\theta}}$. The indicator function being measurable, the only quantity left to compute is the conditional expectation of $\sigma_{\bar{\theta}_0}$ given $\mathcal{T}^{\hat{\theta}}$. Thanks to Lemma 2.28, the fact that $\hat{\theta} > 0$ and the homogeneity property, we get:

$$\mathbb{N}^\psi[\Theta_h \geq \theta|A = \theta_0] = \psi'(\hat{\theta})\mathbb{N}^\psi\left[\sigma_{\hat{\theta}} \mathbf{1}_{\{\mathcal{Z}_h^{\hat{\theta}}>0\}}\right] = \psi'(\hat{\theta})\mathbb{N}^{\psi_{\hat{\theta}}}\left[\sigma \mathbf{1}_{\{\mathcal{Z}_h>0\}}\right].$$

Using that $\mathbb{N}^{\psi_{\hat{\theta}}}[\sigma] = 1/\psi'(\hat{\theta})$, which can be deduced from (28), we get:

$$\begin{aligned}
\mathbb{N}^\psi[\Theta_h \geq \theta|A = \theta_0] &= \psi'(\hat{\theta})\mathbb{N}^{\psi_{\hat{\theta}}}[\sigma] - \psi'(\hat{\theta})\mathbb{N}^{\psi_{\hat{\theta}}}\Big[\int_0^h \mathcal{Z}_a da \mathbf{1}_{\{\mathcal{Z}_h=0\}}\Big] \\
&= 1 - \psi'(\hat{\theta}) \int_0^h da \lim_{\lambda\to\infty} \mathbb{N}^{\psi_{\hat{\theta}}}\left[\mathcal{Z}_a\, \mathrm{e}^{-\lambda \mathcal{Z}_h}\right].
\end{aligned}$$

Now, conditioning by $\mathcal{Z}_a$ and using $\lim_{\lambda\to\infty} u^{\hat{\theta}}(h-t,\lambda) = b_h^{\hat{\theta}}(t)$ as well as (26), we get:

$$\lim_{\lambda\to\infty} \mathbb{N}^{\psi_{\hat{\theta}}}\left[\mathcal{Z}_a\, \mathrm{e}^{-\lambda\mathcal{Z}_h}\right] = \lim_{\lambda\to\infty} \mathbb{N}^{\psi_{\hat{\theta}}}\left[\mathcal{Z}_a\, \mathrm{e}^{-\mathcal{Z}_a u^{\hat{\theta}}(h-a,\lambda)}\right] = \mathbb{N}^{\psi_{\hat{\theta}}}[\mathcal{Z}_a\, \mathrm{e}^{-\mathcal{Z}_a b_h^{\hat{\theta}}(a)}] = \partial_\lambda u^{\hat{\theta}}(s, b_h^{\hat{\theta}}(a)).$$

Then use (49) to get:

$$\begin{aligned}
\int_0^h da \lim_{\lambda\to\infty} \mathbb{N}^{\psi_{\hat{\theta}}}\left[\mathcal{Z}_a\, \mathrm{e}^{-\lambda\mathcal{Z}_h}\right] = \int_0^h da\ \partial_\lambda u^{\hat{\theta}}(s, b_h^{\hat{\theta}}(a)) &= \psi_{\hat{\theta}}(b^{\hat{\theta}}(h)) \int_0^h da\ \frac{|\partial_h b^{\hat{\theta}}(h-a)|}{\psi_{\hat{\theta}}(b^{\hat{\theta}}(h-a))^2} \\
&= \psi_{\hat{\theta}}(b^{\hat{\theta}}(h)) \int_{b^{\hat{\theta}}(h)}^{+\infty} \frac{dr}{\psi_{\hat{\theta}}(r)^2},
\end{aligned}$$

and thus deduce the first equality of the Proposition. Notice $\int^{+\infty} dr/\psi_\theta(r)^2 < +\infty$ thanks to Assumption 2 (in fact this is true in general). Let θ go down to θ_0 and use the fact that $\mathbb{N}^\psi$-a.e. $A \leq \Theta_h$ to get the second equality. □

Remark 4.4. In the quadratic case $\psi(u) = \beta u^2$, we can obtain closed formulae. For all $\theta > 0$, we have:

$$u^\theta(t,\lambda) = \frac{2\theta\lambda}{(2\theta+\lambda)\exp(2\beta\theta t) - \lambda} \quad \text{and} \quad b^\theta(t) = \frac{2\theta}{\mathrm{e}^{2\beta\theta t} - 1}.$$

We have the following exact expression of the conditional distribution for $\theta_0 < \theta$, $\theta_0 < 0$ and with $\bar{\theta}_0 = |\theta_0| = -\theta_0$ and $\hat{\theta} = \theta + 2\,|\theta_0|$:

$$\mathbb{N}^\psi[\Theta_h \geq \theta | A = \theta_0] = 1 + (\beta\hat{\theta}h)/\sinh^2(\beta\hat{\theta}h) - \operatorname{cotanh}(\beta\hat{\theta}h),$$
$$\mathbb{N}^\psi[\Theta_h = A | A = \theta_0] = \beta\theta_0 h/\sinh^2(\beta\theta_0 h) - \operatorname{cotanh}(\beta\theta_0 h).$$

Notice that $\lim_{\theta_0 \to -\infty} \mathbb{N}^\psi[\Theta_h = A | A = \theta_0] = 1$. This correspond to the fact that if A is large, then the tree $\mathcal{T}_A$ is small and has little chance to cross level h. (Notice that $\mathcal{T}_A$ has finite height but $\mathcal{T}_{A-}$ has infinite height.) Thus the time Θ_h is equal to the time when an infinite tree is grafted, that is to the ascension time A.

4.2. **Distribution of the tree at the exit time Θ_h.** Before stating the theorem describing the tree before it overshoots a given height $h > 0$ under the form of a spinal decomposition, we shall explain how this spine is distributed. Recall (50) for the definition of γ_θ.

Lemma 4.5. *Let $\theta \in \Theta^\psi$. The non-negative function*

$$f : t \mapsto \gamma_\theta(b_h^\theta(t)) \exp\Big(-\int_0^t \gamma_\theta(b_h^\theta(r))dr\Big) \tag{56}$$

is a probability density on $[0, h)$ with respect to Lebesgue measure. If ξ is a random variable whose distribution is f, then we have $\mathbb{E}[\exp(-\psi'(\theta)\xi)] < +\infty$.

Notice the integrability property on ξ is trivial if $\theta \geq 0$.

Proof. Notice that $f = g' \mathrm{e}^{-g}$ with $g(t) = \int_0^t \gamma_\theta(b_h^\theta(r))\ dr$. Thus we have $\int_0^h f = \int_0^h g' \mathrm{e}^{-g} = \mathrm{e}^{-g(0)} - \mathrm{e}^{-g(h)}$ and f is a density if and only if $g(h) = \infty$. We deduce from (54) that $\int_0^t \gamma_\theta(b_h^\theta(r))dr$ diverges as t goes to h. The last part of Proposition 4.2 implies that $\mathrm{e}^{-\psi'(\theta)\xi}$ is integrable. □

Recall equation (6) defining the grafting procedure.

Theorem 4.6. *Let $\theta_\infty < \theta$ and let F be a non-negative measurable functional on $\mathbb{T}^2$. Then, we have:*

$$\mathbb{N}^\psi\left[F(\mathcal{T}_{\Theta_h}\ ;\ \mathcal{T}_{\Theta_h -}) | \Theta_h = \theta\right] = \frac{1}{\mathbf{E}\left[\mathrm{e}^{-\psi'(\theta)H_{\mathbf{x}}}\right]}\,\mathbf{E}\left[F\Big([\![\emptyset, \mathbf{x}]\!] \circledast_{i\in I} (\mathcal{T}^i, x_i)\ ;\ ([\![\emptyset, \mathbf{x}]\!] \circledast_{i\in I} (\mathcal{T}^i, x_i)) \circledast (T, \mathbf{x})\Big)\ \mathrm{e}^{-\psi'(\theta)H_{\mathbf{x}}}\right],$$

where the spine $[\![\emptyset, \mathbf{x}]\!]$ is identified with the interval $[0, H_{\mathbf{x}}]$ (and thus $y \in [\![\emptyset, \mathbf{x}]\!]$ is identified with H_y) and:

- *The random variable $H_{\mathbf{x}}$ is distributed with density given by (56).*
- *Conditionally on $H_{\mathbf{x}}$, sub-trees are grafted on the spine $[0, H_{\mathbf{x}}]$ according to a Poisson point measure $\mathcal{N} = \sum_{i\in I} \delta_{(x_i, \mathcal{T}^i)}$ on $[0, H_{\mathbf{x}}] \times \mathbb{T}$ with intensity:*

$$\nu_\theta(da, d\mathcal{T}) = da\ \Big(2\beta(\theta + b_h^\theta(a))\mathbb{N}^{\psi_\theta}[d\mathcal{T}, H_{max}(\mathcal{T}) < h - a] + \int_{(0,+\infty)} r\Pi_{\theta + b_h^\theta(x)}(dr)\mathbb{P}_r^{\psi_\theta}(d\mathcal{T}, H_{max}(\mathcal{T}) < h - a)\Big). \tag{57}$$

- *Conditionally on $H_{\mathbf{x}}$ and on $\mathcal{N}$, T is a random variable on $\mathbb{T}$ with distribution*
$$\mathbf{N}^{\psi_\theta}[dT | H_{max}(T) > h - H_{\mathbf{x}}].$$

In other words, conditionally on $\{\Theta_h = \theta\}$, we can describe the tree before overshooting height h by a spinal decomposition along the ancestral branch of the point at which the overshooting sub-tree is grafted. Conditionally on the height of this point, the overshooting tree has distribution $\mathbf{N}^{\psi_\theta}[dT]$, conditioned on overshooting.

If $\theta > 0$ then $\psi'(\theta) > 0$, and we can understand the weight $\mathrm{e}^{-\psi'(\theta)H_{\mathbf{x}}} / \mathbf{E}\left[\mathrm{e}^{-\psi'(\theta)H_{\mathbf{x}}}\right]$ as a conditioning of the random variable $H_{\mathbf{x}}$ to be larger than an independent exponential random variable with parameter $\psi'(\theta)$.

Remark 4.7. When h goes to infinity, we have, for $\theta \geq 0$, $\lim_{h\to+\infty} b^\theta(h) = 0$ and thus the distribution of Θ_h concentrates on $\Theta^\psi \cap (-\infty, 0)$. For $\theta < 0$ and $\theta \in \Theta_h$, we deduce from (51) that $\lim_{h\to+\infty} b^\theta(h) = \bar\theta - \theta > 0$. And the distribution of ξ in Lemma 4.5 clearly converges to the exponential distribution with parameter $\gamma_\theta(b^\theta(+\infty)) = \psi'(\bar\theta) - \psi(\theta)$. Then the weight $\mathrm{e}^{-\psi'(\theta)H_{\mathbf{x}}} / \mathbf{E}\left[\mathrm{e}^{-\psi'(\theta)H_{\mathbf{x}}}\right]$ changes this distribution. In the end, $H_{\mathbf{x}}$ is asymptotically distributed as an exponential random variable with parameter $\psi'(\bar\theta)$. Notice this is exactly the distribution of the height of a random leaf taken in $\mathcal{T}_A$, conditionally on $\{A = \theta\}$, see Lemma 7.6 in [4].

Remark 4.8. A direct application of Theorem 4.6 with $F(\mathcal{T}; \mathcal{T}')$ chosen equal to

$$G(\mathcal{T}; \mathcal{T}') = \mathbf{1}_{\{\mathbf{m}^{\mathcal{T}}(\mathcal{T})<+\infty, \mathbf{m}^{\mathcal{T}'}(\mathcal{T}')=+\infty\}}, \tag{58}$$

allows to compute for $\theta < 0$:

$$\mathbb{N}^\psi[A = \Theta_h | \Theta_h = \theta] = \left(\psi'(\bar\theta) - \psi'(\theta)\right) \frac{C(\theta, h)}{\psi'(\bar\theta) - \psi'(\theta) C(\theta, h)},$$

where $C(\theta, h) = \psi'(\bar\theta)\psi_\theta(b^\theta(h)) \int_{b^\theta(h)}^{+\infty} dr \;\; \psi_\theta(r)^{-2} = \mathbb{N}^\psi[A = \Theta_h | A = \theta]$. The last equality is a consequence of (55). As $\lim_{h\to+\infty} \mathbb{N}^\psi[A = \Theta_h | A = \theta] = 1$, we get that

$$\lim_{h\to+\infty} \mathbb{N}^\psi[A = \Theta_h | \Theta_h = \theta] = 1.$$

Remark 4.9. By considering the function G in (58) instead of F in the proof of Theorem 4.6, we can recover the distribution of $\mathcal{T}_A$ given in [4], but we also can get the joint distribution of $(\mathcal{T}_{A-}, \mathcal{T}_A)$. Roughly speaking (and unsurprisingly), conditionally on $\{A = \theta\}$, $\mathcal{T}_{A-}$ is obtained from $\mathcal{T}_A$ by grafting an independent random tree T on a independent leaf x chosen according to $\mathbf{m}^{\mathcal{T}_A}(dx)$ and the distribution of T is $\mathbf{N}^{\psi_\theta}[dT,\ H_{max}(T) = +\infty]$. Notice that choosing a leaf at random on $\mathcal{T}_A$ gives that the distribution of $\mathcal{T}_A$ is a size-biased distribution of $\mathbb{N}^{\psi_\theta}[d\mathcal{T}]$.

Proof of Theorem 4.6. Thanks to the compensation formula (45), we can write, if g is any measurable functional $\mathbb{R} \mapsto \mathbb{R}_+$ with support in $(\theta_\infty, +\infty)$:

$$\begin{aligned}
&\mathbb{N}^\psi[F(\mathcal{T}_{\Theta_h} \,;\, \mathcal{T}_{\Theta_h -}) g(\Theta_h)] \\
&\qquad\qquad = \mathbb{N}^\psi\left[\sum_{j\in J} \mathbf{1}_{\{H_{max}(\mathcal{T}_{\theta_j})<h\}} F(\mathcal{T}_{\theta_j} \,;\, \mathcal{T}_{\theta_j} \circledast (\mathcal{T}^j, x_j)) g(\theta_j) \mathbf{1}_{\{H_{x_j} + H_{max}(\mathcal{T}^j) > h\}}\right] \\
&\qquad\qquad = \int_{\Theta^\psi} d\theta \; g(\theta) B(\theta, h),
\end{aligned}$$

where, using the homogeneity property and the Girsanov transformation (32):

$$\begin{aligned}
B(\theta, h) &= \mathbb{N}^\psi\left[\mathbf{1}_{\{H_{max}(\mathcal{T}_\theta)<h\}} \int \mathbf{m}^{\mathcal{T}_\theta}(dx) \int \mathbf{N}^{\psi_\theta}[dT] F(\mathcal{T}_\theta \,;\, \mathcal{T}_\theta \circledast (T, x)) \mathbf{1}_{\{H_x + H_{max}(T) > h\}}\right] \\
&= \mathbb{N}^{\psi_\theta}\left[\mathbf{1}_{\{H_{max}(\mathcal{T})<h\}} \int \mathbf{m}^{\mathcal{T}}(dx) \int \mathbf{N}^{\psi_\theta}[dT] F(\mathcal{T} \,;\, \mathcal{T} \circledast (T, x)) \mathbf{1}_{\{H_x + H_{max}(T) > h\}}\right] \\
&= \mathbb{N}^{\psi_{\bar\theta}}\left[\mathbf{1}_{\{H_{max}(\mathcal{T})<h\}} \int \mathbf{m}^{\mathcal{T}}(dx) \int \mathbf{N}^{\psi_\theta}[dT] F(\mathcal{T} \,;\, \mathcal{T} \circledast (T, x)) \mathbf{1}_{\{H_x + H_{max}(T) > h\}}\right].
\end{aligned}$$

Notice we replaced only $\mathbb{N}^{\psi_\theta}$ by $\mathbb{N}^{\psi_{\bar\theta}}$ in the last equality.

We explain how the term $\mathbf{1}_{\{H_{max}(\mathcal{T})<h\}}$ change the decomposition of $\mathcal{T}$ according to the spine given in Theorem 2.19. Let Φ a non-negative measurable function defined on $[0,+\infty)\times\mathbb{T}$ and φ a non-negative measurable function defined on $[0,+\infty)$. Using Theorem 2.19 and notations therein, we get:

$$\begin{aligned}
&\mathbb{N}^{\psi_{\bar\theta}}\left[\int \mathbf{m}^{\mathcal{T}}(dx)\varphi(H_x)\,\mathrm{e}^{-\langle \mathcal{M}_x,\Phi\rangle}\,\mathbf{1}_{\{H_{max}(\mathcal{T})<h\}}\right]\\
&\qquad= \int_0^\infty da\;\varphi(a)\,\mathrm{e}^{-\psi'_{\bar\theta}(0)a}\;\mathbb{E}\left[\mathrm{e}^{-\sum_{i\in I}\mathbf{1}_{\{z_i\leq a\}}\Phi(z_i,\bar{\mathcal{T}}^i)}\prod_{i\in I, z_i\leq a}\mathbf{1}_{\{H_{max}(\bar{\mathcal{T}}^i)<h-z_i\}}\right]\\
&\qquad= \int_0^h da\;\varphi(a)\;\exp\left(-\psi'(\bar\theta)a-\int_0^a dx\;\mathbf{N}^{\psi_{\bar\theta}}\left[1-\mathrm{e}^{-\Phi(x,\mathcal{T})}\,\mathbf{1}_{\{H_{max}(\mathcal{T})<h-x\}}\right]\right).
\end{aligned}$$

Using the definition of $\mathbf{N}^{\psi_{\bar\theta}}$, see (44), (50) and the Girsanov transformation (32), we get:

$$\begin{aligned}
&\mathbf{N}^{\psi_{\bar\theta}}\left[1-\mathrm{e}^{-\Phi(x,\mathcal{T})}\,\mathbf{1}_{\{H_{max}(\mathcal{T})<h-x\}}\right]\\
&\qquad= \gamma_{\bar\theta}\left(\mathbb{N}^{\psi_{\bar\theta}}\left[1-\mathrm{e}^{-\Phi(x,\mathcal{T})}\,\mathbf{1}_{\{H_{max}(\mathcal{T})<h-x\}}\right]\right)\\
&\qquad= \gamma_{\bar\theta}\left(b^{\bar\theta}(h-x)+\mathbb{N}^{\psi_\theta}\left[\left(1-\mathrm{e}^{-\Phi(x,\mathcal{T})}\right)\mathbf{1}_{\{H_{max}(\mathcal{T})<h-x\}}\right]\right).
\end{aligned}$$

Thanks to (50) and (51), we have for $\lambda\geq 0$:

$$\gamma_{\bar\theta}(b^{\bar\theta}(h-x)+\lambda)=\gamma_{\theta+b^\theta(h-x)}(\lambda)+\gamma_\theta(b^\theta(h-x))+\psi'(\theta)-\psi'(\bar\theta).$$

Take $\lambda=\mathbb{N}^{\psi_\theta}\left[\left(1-\mathrm{e}^{-\Phi(x,\mathcal{T})}\right)\mathbf{1}_{\{H_{max}(\mathcal{T})<h-x\}}\right]$, to deduce that:

$$\begin{aligned}
&\mathbb{N}^{\psi_{\bar\theta}}\left[\int \mathbf{m}^{\mathcal{T}}(dx)\varphi(H_x)\,\mathrm{e}^{-\langle \mathcal{M}_x,\Phi\rangle}\,\mathbf{1}_{\{H_{max}(\mathcal{T})<h\}}\right]\\
&\qquad= \int_0^h da\;\varphi(a)\;\exp\left(-\psi'(\theta)a-\int_0^a dx\;\gamma_\theta(b^\theta(h-x))\right)\\
&\qquad\qquad\exp\left(-\int_0^a dx\;\gamma_{\theta+b^\theta(h-x)}\left(\mathbb{N}^{\psi_\theta}\left[\left(1-\mathrm{e}^{-\Phi(x,\mathcal{T})}\right)\mathbf{1}_{\{H_{max}(\mathcal{T})<h-x\}}\right]\right)\right)\\
&\qquad= \int_0^h da\;\varphi(a)\;\exp\left(-\psi'(\theta)a-\int_0^a dx\;\gamma_\theta(b^\theta(h-x))\right)\mathbb{E}\left[\mathrm{e}^{-\sum_{i\in I}\mathbf{1}_{\{z_i\leq a\}}\Phi(z_i,\tilde{\mathcal{T}}^i)}\right],
\end{aligned}$$

where under $\mathbb{E}$, $\sum_{i\in I}\delta_{(z_i,\tilde{\mathcal{T}}^i)}(dz,d\mathcal{T})$ is a Poisson point measure on $[0,h]\times\mathbb{T}$ with intensity ν_θ in (57). Since Laplace transforms characterize random measure distributions, we get that for any non-negative measurable function $\tilde F$, we have:

$$\begin{aligned}
&\mathbb{N}^{\psi_{\bar\theta}}\left[\int \mathbf{m}^{\mathcal{T}}(dx)\tilde F(H_x,\mathcal{M}_x)\mathbf{1}_{\{H_{max}(\mathcal{T})<h\}}\right]\\
&\qquad= \int_0^h da\;\mathrm{e}^{-\psi'(\theta)a-\int_0^a dx\;\gamma_\theta(b^\theta(h-x))}\,\mathbb{E}\left[\tilde F\left(a,\sum_{i\in I}\mathbf{1}_{\{z_i\leq a\}}\delta_{(z_i,\tilde{\mathcal{T}}^i)}\right)\right].
\end{aligned}$$

If we identify the spine $[\![\emptyset,x]\!]$ (with its metric) to the interval $[0,H_x]$ (with the Euclidean metric), we can use this result to compute $B(\theta,h)$ with:

$$\tilde F(H_x,\mathcal{M}_x)=\int \mathbf{N}^{\psi_\theta}[dT\mid H_x+H_{max}(T)>h]F(\mathcal{T}\;;\;\mathcal{T}\circledast(T,x)),$$

$\mathcal{M}_x = \sum_{i\in I_x} \delta_{(H_{x_i},\mathcal{T}^i)}$ and $\mathcal{T} = [0, H_x] \circledast_{i\in I_x} (\mathcal{T}^i, H_{x_i})$. Since $\mathbf{N}^{\psi_\theta}[H_{max}(\mathcal{T}) > h] = \gamma_\theta(b^\theta(h))$, we have:

$$\gamma_\theta(b^\theta(h - H_x))\tilde{F}(H_x, \mathcal{M}_x) = \int \mathbf{N}^{\psi_\theta}[dT]F(\mathcal{T}\ ;\ \mathcal{T} \circledast (T, x))\mathbf{1}_{\{H_x+H_{max}(T)>h\}}.$$

Therefore, we have:

$$\begin{aligned} B(\theta, h) &= \mathbb{N}^{\psi_{\bar\theta}}\left[\mathbf{1}_{\{H_{max}(\mathcal{T})<h\}} \int \mathbf{m}^{\mathcal{T}}(dx) \int \mathbf{N}^{\psi_\theta}[dT]F(\mathcal{T}\ ;\ \mathcal{T} \circledast (T, x))\mathbf{1}_{\{H_x+H_{max}(T)>h\}}\right] \\ &= \int_0^h da\ \gamma_\theta(b^\theta(h - a))\, \mathrm{e}^{-\psi'(\theta)a - \int_0^a dx\ \gamma_\theta(b^\theta(h-x))}\, \mathbb{E}\left[\tilde{F}\left(a, \sum_{i\in I} \mathbf{1}_{\{z_i\le a\}}\delta_{(z_i,\tilde{\mathcal{T}}^i)}\right)\right]. \end{aligned}$$

Thus, we get:

$$\begin{aligned} &\mathbb{N}^\psi[F(\mathcal{T}_{\Theta_h}\ ;\ \mathcal{T}_{\Theta_h-})g(\Theta_h)] \\ &\qquad = \int_{\Theta^\psi} d\theta\ g(\theta) \int_0^h da\ \gamma_\theta(b^\theta(h - a))\, \mathrm{e}^{-\psi'(\theta)a - \int_0^a dx\ \gamma_\theta(b^\theta(h-x))} \\ &\qquad\qquad \mathbb{E}\left[\tilde{F}\left(a, \sum_{i\in I} \mathbf{1}_{\{z_i\le a\}}\delta_{(z_i,\tilde{\mathcal{T}}^i)}\right)\right]. \end{aligned}$$

Then use the distribution of Θ_h under $\mathbb{N}^\psi$ given in Proposition 4.2 to conclude. □

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