An Active Curve Approach for Tomographic Reconstruction of Binary Radially Symmetric Objects

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Abstract This paper deals with tomographic reconstruction of radially symmetric objects from a single radiograph, in order to study the behavior of shocked material. Usual tomographic reconstruction algorithms (such as generalized inverse or filtered back-projection) cannot be applied here. To improve the reconstruction, we assume that the object is binary so that it may be described by curves that separate the two materials. We present a BV-model that leads to a non local Hamilton-Jacobi equation, via a level set strategy.

1 Introduction

We are interested here in a very specific application of tomographic reconstruction for a physical experiment which goal is to study the behavior of a material under a shock. During the deformation of the object, we obtain an X-ray radiography by high speed image capture. We suppose that the object is radially symmetric, so that one radiograph is enough to reconstruct the 3D object.

Physicists are looking for the shape of the interior at some fixed interest time. At that time, the interior may be composed of several holes which also may be very irregular. We deal here with a synthetic object that contains all the standard difficulties that may appear (see Fig. 1). These difficulties are characterized by:

- Several disconnected holes.
- A small hole located on the symmetry axis (which is the area where the details are difficult to recover).

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- Smaller and smaller details on the boundary of the top hole in order to determine a lower bound detection.

Our framework is completely different from the usual tomographic point of view, and usual techniques (such as filtered back-projection) are not adapted to our case.

2 A Continuous Model in BV Space

Let us explicit the projection operator involved in the tomography process. This operator, denoted by H, is given, for every function $f \in L^{\infty}(\mathbb{R}_{+} \times \mathbb{R})$ with compact support, by

$$\forall (u,v) \in \mathbb{R} \times \mathbb{R}, \qquad Hf(u,v) = 2\int_{|u|}^{+\infty} f(r,v)\frac{r}{\sqrt{r^2-u^2}}dr. \tag{1}$$

For more details one can refer to [1]. Similarly the adjoint operator H^* of H is

$$\forall (r,z) \in \mathbb{R} \times \mathbb{R}, \qquad H^*g(r,z) = 2\int_{0}^{|r|} g(u,z)\frac{|r|}{\sqrt{r^2-u^2}}du. \tag{2}$$

Thanks to the symmetry, this operator characterizes the Radon transform of the object and so is invertible. However, the operator H^{-1} is not continuous with respect to the suitable topologies. Consequently, a small variation on the measure g leads to significant errors on the reconstruction. As radiographs are strongly perturbed, applying H^{-1} to data leads to a poor reconstruction. Due to the experimental setup there are two main perturbations:

- A blur, due to the detector response and the X-ray source spot size. We denote by F the effect of blurs and consider the simplified case where F is supposed to be linear.
- A noise which is supposed for simplicity to be an additive Gaussian white noise of mean 0, denoted by ε.

Consequently, the projection of the object f will be $g = F(Hf) + \varepsilon$. The comparison between the theoretical projection Hf and the perturbed one is shown in Fig. 1. The reconstruction using the inverse operator H^{-1} applied to g is given by Fig. 2. It is clear from Fig. 2 that the use of the inverse operator is not suitable. In what follows, we will call "experimental data " the image which corresponds to the blurred projection of a "fictive" object of density 0 with some holes of known "density" $\lambda > 0$. Consequently, the space of admissible objects will be the set $\mathscr{F}$ of functions f that take values in $\{0,\lambda\}$. Such functions $f \in \mathscr{F}$ are defined on $\mathbb{R}^2$ and have compact support included in some open bounded subset of $\mathbb{R}^2$, say Ω. Note that $\mathscr{F}$ is a subspace of the bounded variation functions space

$$BV(\Omega) = \{u \in L^1(\Omega) \mid J(u) < +\infty\}$$

Fig. 1 Synthetic object, theoretical projection Hf, real projection with noise and blur.

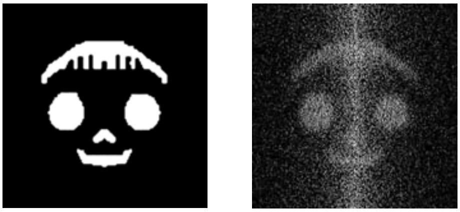

Fig. 2 Comparison between the real object on the left-hand side and the reconstruction computed with H^{-1} applied to the real projection on the right-hand side.

where $J(u)$ stands for the total variation of u (see [3] for example). Furthermore, a function f of $\mathscr{F}$ is characterized by the knowledge of the curves that limit the two areas where f is equal to λ and to 0. Indeed, as the support of the function f is bounded, these curves are disjoint Jordan curves and the inside density λ whereas the outside one is 0. We use here a standard technique for image reconstruction: we introduce an energy functional that is a function of γ_f where γ_f is a set of disjoint Jordan curves. We may split this energy in two terms:

- The first one is a matching term. It is the usual L^2-norm between $F(Hf)$ and the data g (where H is given by (1)), so that $E_1(\gamma_f) = \frac{1}{2}\|F(Hf) - g\|_2^2$. Here $\|\cdot\|_2$ stands for the $L^2(\Omega)$- norm and $g \in L^2(\Omega)$.
- The second term is a regularization term: $E_2(\gamma_f) = \ell(\gamma_f)$ where $\ell(\gamma_f)$ denotes the length of the curve γ_f.This penalization term may be also viewed as the total variation $J(f)$ (up to a multiplicative constant) of the function f because of the binarity.

Hence the total energy functional is

$$E(\gamma_f) = \frac{1}{2}\|F(Hf) - g\|_2^2 + \alpha \ell(\gamma_f) \tag{3}$$

which is an adaptation of the well-known Mumford-Shah energy functional introduced in [4]. The "optimal" value of $\alpha > 0$ may depend on the data.
Therefore, we consider the following minimization problem

$$(\mathscr{P}) \qquad \min \ \{\, E(\gamma_f) \mid f \in BV(\Omega),\ f(x) \in \{0, \lambda\} \text{ a.e. on } \Omega \},$$

and we first get an existence result:

Theorem 1. *Problem $(\mathscr{P})$ admits at least one solution.*

Proof. See Theorem 1 of [2] □

3 Computation of the Energy Derivative

Now, we are interested in optimality conditions. In what follows, for mathematical reasons, we have to add an extra assumption: the curves γ_f are $\mathscr{C}^1$ so that the normal vector of the curves is well-defined (as an orthogonal vector to the tangent one). Unfortunately, with this assumption, we cannot compute easily the derivative of the energy in the $BV(\Omega)$ framework. Indeed we need regular curves and we do not know if the $BV(\Omega)$ minimizer provides a curve with the required regularity. Moreover, the set of constraints is not convex and it is not easy to compute the Gateaux derivative (no admissible test functions).

So we have few hope to get classical optimality conditions and we rather compute minimizing sequences. We focus on those that are given via the gradient descent method inspired by [4]. Formally, we look for a family of curves $(\gamma_t)_t \geq 0$ such that $\dfrac{\partial E}{\partial \gamma}(\gamma_t) \leq 0$ that is: $E(\gamma_t)$ decreases as $t \to +\infty$. Let us compute the energy variation when we operate a small deformation on the curves γ. In other words, we will compute the energy Gâteau derivative for a small deformation $\delta\gamma$:

$$\frac{\partial E}{\partial \gamma}(\gamma) \cdot \delta\gamma = \lim_{t \to 0} \frac{E(\gamma + t\delta\gamma) - E(\gamma)}{t}.$$

We first focus on local deformations $\delta\gamma$. Let (r_0, z_0) be a point P of γ. We consider a local reference system which center is P and axis are given by the tangent and normal vectors at P. We denote (ξ, η) the new generic coordinates in this reference system and still denote $f(\xi, \eta) = f(r, z)$ for convenience. We apply the implicit functions theorem to parametrize the curve: there exist a neighborhood U of P and a $\mathscr{C}^1$ function h such that, for every $(\xi, \eta) \in U$,

$$(\xi, \eta) \in \gamma \iff \eta = h(\xi).$$

Eventually, we get a neighborhood U of P, a neighborhood I of ξ_0 and a $\mathscr{C}^1$ function h such that

$$\gamma \cap U = \left\{ (\xi, \eta) \in \mathbb{R}^2 \mid \eta = h(\xi),\ \xi \in I \right\}.$$

The local parametrization is oriented along the outward normal $\mathbf{n}$ to the curve γ at point P. More precisely, we define the local coordinate system $(\tau, \mathbf{n})$ where τ is the usual tangent vector, $\mathbf{n}$ is the direct orthonormal vector; we set the curve orientation so that $\mathbf{n}$ is the outward normal. The function f if then defined on U by

$$f(\xi,\eta) = \begin{cases} \lambda \text{ if } \eta < h(\xi) \ , \\ 0 \text{ if } \eta \geq h(\xi) \ . \end{cases}$$

We then consider a local (limited to U) deformation $\delta\gamma$ along the normal vector. This is equivalent to handling a $\mathscr{C}^1$ function δh whose support is included in I. The new curve γ_t obtained after the deformation $t\delta\gamma$ is then parametrized by

$$\eta = \begin{cases} h(\xi) + t\delta h(\xi) \text{ for } (\xi,\eta) \in U \ , \\ \gamma \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

This defines a new function f_t:

$$f_t(\xi,\eta) = \begin{cases} f(\xi,\eta) \text{ if } (\xi,\eta) \notin U \ , \\ \lambda \qquad\quad \text{if } (\xi,\eta) \in U \cap \{\eta < h(\xi) + t\delta h(\xi)\} \ , \\ 0 \qquad\quad \text{if } (\xi,\eta) \in U \cap \{\eta \geq h(\xi) + t\delta h(\xi)\} \ . \end{cases} \tag{4}$$

We will also set $\delta f_t = f_t - f$. This deformation is described in Fig. 3.

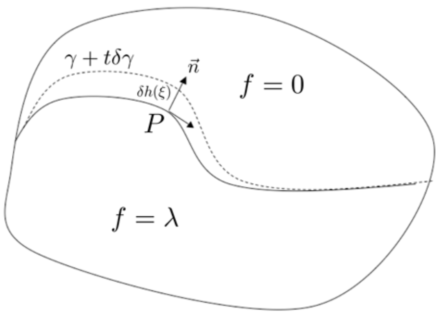

Fig. 3 Description of a local deformation of the initial curve γ. P is the current point, U is the neighborhood of P in which the deformation is restricted to and $\gamma + t\delta\gamma$ is the new curve after deformation. The interior of the curve is the set where $f = \lambda$.

The energy E_2 Gâteau derivative has already been computed in [4] and is

$$\frac{\partial E_2}{\partial \gamma}(\gamma)\delta\gamma = -\int_\gamma \text{curv}(\gamma)\big(\xi, h(\xi)\big)\delta h(\xi) d\xi \ ,$$

where curv denotes the curvature of the curve and δh is the parametrization of $\delta\gamma$. It remains to compute the derivative of E_1 . We first estimate δf_t: a simple computation shows that

$$\delta f_t(\xi,\eta) = \begin{cases} 0 \text{ if } \eta \geq h(\xi) + t\delta h(\xi) \text{ or } \eta(\xi) \leq h(\xi) \ , \\ \lambda \text{ if } h(\xi) \leq \eta \leq h(\xi) + t\delta h(\xi) \ , \end{cases} \quad \text{in case } \delta h \geq 0 \ ,$$

and

$$\delta f_t(\xi,\eta) = \begin{cases} 0 \text{ if } \eta \le h(\xi)+t\delta h(\xi) \text{ or } \eta(\xi) \ge h(\xi)\,, \\ -\lambda \text{ if } h(\xi) \ge \eta \ge h(\xi)+t\delta h(\xi)\,, \end{cases} \text{ in case } \delta h \le 0\,.$$

Now we compute $E_1(\gamma_t) - E_1(\gamma)$ where γ (resp. γ_t) is the curve associated to the function f (resp. f_t):

$$\begin{aligned}
&E_1(\gamma_t) - E_1(\gamma) \\
&\quad = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \left((g - FHf_t)^2 - (g - FHf)^2\right)(u,v)\,du\,dv \\
&\quad = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \left((g - FHf - FH\delta f_t)^2 - (g - FHf)^2\right)(u,v)\,du\,dv \\
&\quad = -\int_{\mathbb{R}}\int_{\mathbb{R}} (g - FHf)(u,v)\,FH\delta f_t(u,v)\,du\,dv + \underbrace{\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (FH\delta f_t)^2(u,v)\,du\,dv}_{=o(t)} \\
&\quad = -\langle (g - FHf), FH\delta f_t\rangle_{L^2} + o(t) \\
&\quad = -\langle H^*F^*(g - FHf), \delta f_t\rangle_{L^2} + o(t).
\end{aligned}$$

To simplify the notations, we denote by $\mathscr{A}f := (H^*Fg - H^*FFHf)$ so that we need to compute

$$\lim_{t\to 0}\frac{1}{t}\langle \mathscr{A}f, \delta f_t\rangle_{L^2}.$$

As δf_t is zero out of the neighbourhood U, we have

$$\langle \mathscr{A}f, \delta f_t\rangle_{L^2} = \int_U \mathscr{A}f(\xi,\eta)\delta f_t(\xi,\eta)\,d\xi\,d\eta.$$

In the case $\delta h \ge 0$, we have,

$$\langle \mathscr{A}f, \delta f_t\rangle_{L^2} = \lambda \int_{\xi\in\gamma}\int_{\eta=h(\xi)}^{\eta=h(\xi)+t\delta h(\xi)} \mathscr{A}f(\xi,\eta)\,d\xi\,d\eta.$$

As the function $\mathscr{A}f$ is continuous (and thus bounded on U), we may pass to the limit by dominated convergence and get

$$\lim_{t\to 0}\frac{1}{t}\langle \mathscr{A}f, \delta f_t\rangle_{L^2} = \lambda \int_\gamma \mathscr{A}f\big(\xi, h(\xi)\big)\delta h(\xi)\,d\xi.$$

In the case $\delta h < 0$, we have

$$\langle \mathscr{A}f, \delta f_t\rangle_{L^2} = \int (-\lambda) \int_{\xi\in\gamma}\int_{\eta=h(\xi)+t\delta h(\xi)}^{\eta=h(\xi)} \mathscr{A}f(\xi,\eta)\,d\xi\,d\eta$$

and we obtain the same limit as in the non negative case. Finally, the energy derivative is

$$\frac{\partial E}{\partial \gamma}(\gamma_t)\cdot\delta\gamma_t = -\int_\gamma \left(\lambda\mathscr{A}f + \alpha\,\mathrm{curv}(\gamma_t)(\xi,h(\xi))\right)\delta h(\xi)\,d\xi\,. \tag{5}$$

As $\delta h = \langle \delta \gamma_t, \mathbf{n} \rangle$ formula (5) may be written

$$\frac{\partial E}{\partial \gamma}(\gamma_t) \cdot \delta \gamma_t = -\int_\gamma \lambda \left(\lambda \mathscr{A} f + \alpha \text{curv}(\gamma_t)(s)\right) \langle \delta \gamma_t, \mathbf{n} \rangle \, ds \tag{6}$$

where $\mathbf{n}$ denotes the outward pointing normal unit vector of the curve γ, $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^2$ and $c(s)$ is a positive coefficient that depends on the curvilinear abscissa s.

The latter expression is linear and continuous in $\delta \gamma$, this formula is also true for a non local deformation (which can be achieved by summing local deformations).

4 Front Propagation and Level Set Method

Now we consider a family of curves $(\gamma_t)_{t \geq 0}$ that converges toward a local minimum of the energy functional. From equation (6), it is clear that if the curves (γ_t) evolve according to the differential equation

$$\frac{\partial \gamma}{\partial t} = (\lambda \mathscr{A} f + \alpha \text{curv}(\gamma_f))\mathbf{n}, \tag{7}$$

the total energy will decrease.

The level set method consists in viewing the curves γ as the 0-level set of a smooth real function ϕ defined on $\mathbb{R}^2$ (see [5]). The function f that we are looking for is then given by

$$f(x) = \lambda 1_{\phi(x) > 0}.$$

Let us write the evolution PDE for functions $\phi_t = \phi(t, \cdot)$ that correspond to the curves γ_t. Let $x(t)$ be a point of the curve γ_t and let us follow that point during the evolution. We know that this point evolves according to equation (7)

$$x'(t) = \left(\lambda \mathscr{A} f + \alpha \text{ curv}(\gamma_f)\right) (x(t))\mathbf{n}.$$

We can rewrite this equation in terms of the function ϕ recognizing that

$$\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|} \quad \text{and} \quad \text{curv}(\gamma) = \text{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)$$

where ∇ stands for the gradient of ϕ with respect to x, $|\cdot|$ denotes the Euclidean norm. The evolution equation becomes

$$x'(t) = \left(\lambda^2 \mathscr{A}\left(1_{\phi(t,\cdot)>0}\right) + \alpha \text{ div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)\right) \frac{\nabla \phi}{|\nabla \phi|}(t, x(t)).$$

Then, as the point $x(t)$ remains on the curve γ_t, it satisfies $\phi_t(x(t)) = \phi(t, x(t)) = 0$. By differentiating this expression, we obtain

$$\frac{\partial \phi}{\partial t} + \langle \nabla \phi, x'(t) \rangle = 0$$

which leads to the following evolution equation for ϕ:

$$\frac{\partial \phi}{\partial t} + |\nabla_x \phi| \left(\lambda^2 \mathscr{A} \left(1_{\phi(t,\cdot)>0} \right) + \alpha \operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) \right) = 0 \, ,$$

that is

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \left(\lambda^2 H^* F F H \left(1_{\phi(t,\cdot)>0} \right) - \alpha \operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) - \lambda H^* F g \right) . \tag{8}$$

The above equation is an Hamilton-Jacobi equation which involves a non local term (through H and F). Such equations are difficult to handle especially when it is not monotone (which is the case here). In particular, existence and/or uniqueness of solutions (even in the viscosity sense) are not clear. Nevertheless, some numerical experiments have been carried out using excplicit schemes based on Sethian's techniques for level-set methods and give interesting results.

Fig. 4 Synthetic object, H^{-1} applied to the real projection , computed solution.

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