

PENALIZATION OF GALTON-WATSON PROCESSES

ROMAIN ABRAHAM AND PIERRE DEBS

ABSTRACT. We apply the penalization technique introduced by Roynette, Vallois, Yor for Brownian motion to Galton-Watson processes with a penalizing function of the form $P(x)s^x$ where P is a polynomial of degree p and $s \in [0, 1]$. We prove that the limiting martingales obtained by this method are most of the time classical ones, except in the super-critical case for $s = 1$ (or $s \rightarrow 1$) where we obtain new martingales. If we make a change of probability measure with this martingale, we obtain a multi-type Galton-Watson tree with p distinguished infinite spines.

1. INTRODUCTION

Let $(Z_n)_{n \geq 0}$ be a Galton-Watson process (GW) associated with an offspring distribution $q = (q_n, n \in \mathbb{N})$. We denote by μ the first moment of q and recall that the process is said to be sub-critical (resp. critical, resp. super-critical) if $\mu < 1$ (resp. $\mu = 1$, resp. $\mu > 1$) and that the process suffers a.s. extinction in the sub-critical and critical cases (unless the degenerate case $q_1 = 1$) whereas it has a positive probability $1 - \kappa$ of survival in the super-critical case. Moreover, the constant κ is the smallest positive fix point of the generating function f of q . We refer to [4] for general results on GW processes.

It is easy to check that the two processes $(Z_n/\mu^n)_{n \geq 0}$ and $(\kappa^{Z_n-1})_{n \geq 0}$ are martingales with respect to the natural filtration $(\mathcal{F}_n)_{n \geq 0}$ associated with $(Z_n)_{n \geq 0}$, with mean 1. Moreover, given a martingale $(M_n)_{n \geq 0}$ with mean 1, we can define a new process $(\tilde{Z}_n)_{n \geq 0}$ by a change of probability: for every nonnegative measurable functional φ , we have

$$\mathbb{E} \left[\varphi(\tilde{Z}_k, 0 \leq k \leq n) \right] = \mathbb{E} [M_n \varphi(Z_k, 0 \leq k \leq n)].$$

The distribution of the process $\tilde{Z}$ and of its genealogical tree is well-known for the two previous martingales. In the sub-critical or critical case, the process associated with the martingale $(Z_n/\mu^n)_{n \geq 0}$ is the so-called sized-biased GW and is a two-type GW. It can also be viewed as a version of the process conditioned on non-extinction, see [11]. The associated genealogical tree is composed of an infinite spine on which are grafted trees distributed as the original one. In the super-critical case, if $\kappa \neq 0$, the process associated with the martingale $(\kappa^{Z_n-1})_{n \geq 0}$ is the original GW conditioned on extinction. It is a sub-critical GW with generating function $\tilde{f}(\cdot) = f(\kappa \cdot)/\kappa$ and mean $\tilde{\mu} = f'(\kappa)$. By combining these two results, we get a third martingale namely

$$(1) \quad M_n^{(1)} = \frac{Z_n \kappa^{Z_n-1}}{f'(\kappa)^n}$$

and the associated process $\tilde{Z}$ is distributed, if $0 < \kappa < 1$, as the size-biased process of the GW conditioned on extinction.

Date: March 27, 2018.

2010 Mathematics Subject Classification. 60J80, 60G42.

Key words and phrases. Galton-Watson trees, penalization, conditioning.

A general method called *penalization* has been introduced by Roynette, Vallois and Yor [15, 14, 16] in the case of the one-dimensional Brownian motion to generate new martingales and to define, by a change of measure, Brownian-like processes conditioned on some specific zero-probability events. This method has also been used for similar problems applied to random walks, [5, 6]. It consists in our case in considering a function $\varphi(n, x)$ and studying the limit

$$(2) \quad \lim_{m \rightarrow +\infty} \frac{\mathbb{E} [\mathbb{1}_{\Lambda_n} \varphi(m+n, Z_{m+n})]}{\mathbb{E} [\varphi(m+n, Z_{m+n})]}$$

with $\Lambda_n \in \mathcal{F}_n$. If this limit exists, it takes the form $\mathbb{E}[\mathbb{1}_{\Lambda_n} M_n]$ where the process $(M_n)_{n \in \mathbb{N}}$ is a positive martingale with $M_0 = 1$ (see [17] for more details).

The study of conditioned GW goes back to the seminal work of Kesten [11] and has recently received a renewed interest, see [9, 2, 1], mainly because of the possibility of getting other types of limiting trees than Kesten's. This work can also be viewed as part of this problem. For instance, penalizing by the martingale Z_n/μ^n prevents the process from extinction (this is the case considered in [11]) whereas considering the weight κ^{Z_n-1} penalizes the paths where the size of the population gets large.

In order to generalize the martingale (1), we first consider the function $\varphi(x) = H_p(x)s^x$ (that does not depend on n) for $0 < s < 1$ where H_p denotes the p -th Hilbert's polynomial defined by

$$(3) \quad H_0(x) = 1 \text{ and } H_p(x) = \frac{1}{p!} \prod_{k=0}^{p-1} (x - k) \text{ for } p \geq 1.$$

We prove that the limit (2) exists for every $s \in [0, 1)$ but we always get already known limiting martingales. More precisely, see Theorems 3.3, 3.5 and 3.8, we have for every $p \in \mathbb{N}$, every $s \in [0, 1)$, every $n \in \mathbb{N}$ and every $\Lambda_n \in \mathcal{F}_n$,

- *Critical and sub-critical case.*

$$\lim_{m \rightarrow +\infty} \frac{\mathbb{E} [\mathbb{1}_{\Lambda_n} H_p(Z_{m+n}) s^{Z_{m+n}}]}{\mathbb{E} [H_p(Z_{m+n}) s^{Z_{m+n}}]} = \begin{cases} \mathbb{E}[\mathbb{1}_{\Lambda_n}] & \text{if } p = 0, \\ \mathbb{E}[Z_n/\mu^n \mathbb{1}_{\Lambda_n}] & \text{if } p \geq 1. \end{cases}$$

This result also holds in the critical case for $s = 1$.

- *Super-critical case.* We set $\mathfrak{a} = \min\{k \geq 0, q_k > 0\}$. We have for every $p \geq 0$,

$$\lim_{m \rightarrow +\infty} \frac{\mathbb{E} [\mathbb{1}_{\Lambda_n} H_p(Z_{m+n}) s^{Z_{m+n}}]}{\mathbb{E} [H_p(Z_{m+n}) s^{Z_{m+n}}]} = \begin{cases} \mathbb{E} [\kappa^{Z_n-1} \mathbb{1}_{\Lambda_n}] & \text{if } p = 0 \text{ and } \mathfrak{a} = 0, \\ \mathbb{E} \left[\frac{Z_n \kappa^{Z_n-1}}{f'(\kappa)^n} \mathbb{1}_{\Lambda_n} \right] & \text{if } p \geq 1 \text{ and } \mathfrak{a} = 0, \\ \mathbb{E} \left[\frac{1}{q_1^n} \mathbb{1}_{Z_n=1} \mathbb{1}_{\Lambda_n} \right] & \text{if } \mathfrak{a} = 1, \\ \mathbb{E} \left[q_{\mathfrak{a}}^{-\frac{\mathfrak{a}^n-1}{\mathfrak{a}-1}} \mathbb{1}_{Z_n=\mathfrak{a}^n} \mathbb{1}_{\Lambda_n} \right] & \text{if } \mathfrak{a} \geq 2. \end{cases}$$

Let us mention that the choice of the Hilbert's polynomials is only here to ease the computations but does not have any influence on the limit. Considering any polynomial of degree p that vanishes at 0 leads to the same limit as for H_p .

A more interesting feature is to consider, in the super-critical case, $s = 1$ or a sequence s_n that tends to 1. It appears that the correct speed of convergence, in order to get non-trivial limits, leads to consider functions of the form

$$\varphi_p(n, x) = H_p(x) e^{-ax/\mu^n}$$

where $a \in \mathbb{R}_+$, see Theorem 4.2. We also describe the genealogical tree of $\tilde{Z}$, see Theorem 4.9, which is the genealogical tree of a non-homogeneous multi-type GW (the offspring distribution of a node depends on its type and its generation). For the tree associated with the function φ_p , the types of the nodes run from 0 to p , the root being of type p . Moreover, the sum of the types of the offspring of one node is equal to the type of this node. Hence, nodes of type 0 give birth to nodes of type 0, nodes of type 1 give birth to one node of type 1 and nodes of type 0, nodes of type 2 give birth to either one node of type 2 or two nodes of type 1, and nodes of type 0, etc. For instance, the figure below gives some possible trees with a root of type 2 or 3. The type of the node is written in it, black nodes are of type 0.

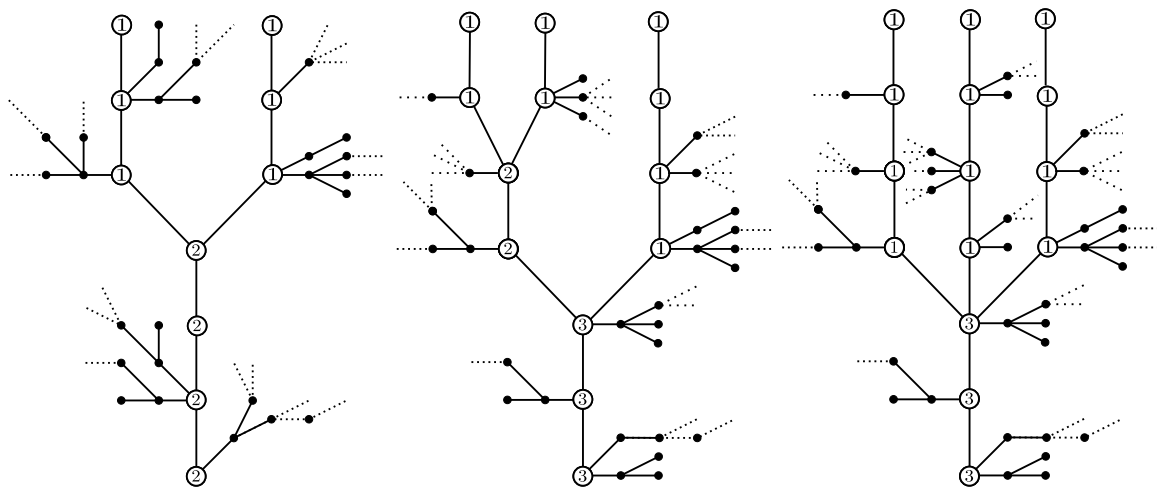


FIGURE 1. Trees with a root of respective type 2, 3 and 3.

We see that, if the root is of type p , the tree exhibits a skeleton with p infinite spines on which are grafted trees of type 0. The distribution of such a tree is given in Definition 4.4. Let us mention that the trees of type 0 already appear in [3] and may be infinite, the p -spines of the skeleton are not the only infinite spines of the tree. Multi-spine trees have already been considered, [13, 8], but they differ from those introduced here.

In the sub-critical case, if we suppose that there exists a second fix point κ for the generating function f , the associated GW can be viewed as a super-critical GW conditioned on extinction and can be obtained from this super-critical GW by a standard change of measure. Then by combining the two changes of measure, the previous results can be used to get similar results in such a sub-critical case, see Theorem 5.1.

The paper is organized as follows. In the second section, we introduce the formalism of discrete trees that we use in all the paper and define the distribution of Galton-Watson trees. In Section 3, we compute all the limits in the case $s \in [0, 1)$. We then compute in Section 4 the limit in the super-critical case when $s_n \rightarrow 1$ and describe the distribution of the modified genealogical tree. We deduce the same kind of results in the sub-critical case in Section 4 and finish with an appendix that contains a technical lemma on Hilbert polynomials that we use in the proofs.

2. NOTATIONS

2.1. The set of discrete trees. Let $\mathcal{U} = \bigcup_{n=0}^{+\infty} (\mathbb{N}^*)^n$ be the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\partial\}$. For every $u \in \mathcal{U}$, we set $|u|$ the length of u i.e. the unique integer $n \in \mathbb{N}$ such that $u \in (\mathbb{N}^*)^n$. If u and v are two sequences of $\mathcal{U}$, we set uv the concatenation of the two sequences with the convention $\partial u = u\partial = u$. For every $u \in \mathcal{U} \setminus \{\partial\}$, we define $\bar{u}$ the unique element of $\mathcal{U}$ such that $u = \bar{u}i$ for some $i \in \mathbb{N}^*$.

A tree $\mathbf{t}$ rooted at $u \in \mathcal{U}$ is a subset of $\mathcal{U}$ that satisfies

- $u \in \mathbf{t}$.
- $\forall v \in \mathcal{U}, |v| < |u| \implies v \notin \mathbf{t}$.
- $\forall v \in \mathbf{t} \setminus \{u\}, \bar{v} \in \mathbf{t}$.
- $\forall v \in \mathbf{t}, \exists k_v(\mathbf{t}) \in \mathbb{N}, \forall i \in \mathbb{N}^*, vi \in \mathbf{t} \iff i \leq k_v(\mathbf{t})$.

We denote by $\mathbb{T}_u$ the set of trees rooted at u and by $\mathbb{T} = \bigcup_{u \in \mathcal{U}} \mathbb{T}_u$ the set of all trees.

For a tree $\mathbf{t} \in \mathbb{T}$, we set $H(\mathbf{t})$ its height:

$$H(\mathbf{t}) = \max\{|u|, u \in \mathbf{t}\},$$

and we denote, for every $h \in \mathbb{N}^*$, by $\mathbb{T}^{(h)}$ (resp. $\mathbb{T}_u^{(h)}$) the subset of trees of $\mathbb{T}$ (resp. $\mathbb{T}_u$) with height less than h .

For every $n \in \mathbb{N}^*$, we denote by $\mathbf{1}_n = (1, \dots, 1) \in (\mathbb{N}^*)^n$, and we write for simplicity $\mathbb{T}_n$ (resp. $\mathbb{T}_n^{(h)}$) instead of $\mathbb{T}_{\mathbf{1}_n}$ (resp. $\mathbb{T}_{\mathbf{1}_n}^{(h)}$).

For every $\mathbf{t} \in \mathbb{T}$ and every $u \in \mathbf{t}$, we set $\mathbf{t}_u$ the subtree of $\mathbf{t}$ rooted at u i.e.

$$\mathbf{t}_u = \{v \in \mathbf{t}, \exists w \in \mathcal{U}, v = uw\}.$$

For every $\mathbf{t} \in \mathbb{T}$ and every $n \in \mathbb{N}$, we denote by $z_n(\mathbf{t})$ the number nodes of $\mathbf{t}$ at height n :

$$z_n(\mathbf{t}) = \text{Card}(\{u \in \mathbf{t}, |u| = n\}).$$

For every $n \in \mathbb{N}^*$, we define on $\mathbb{T}$ the restriction operator r_n by

$$\forall \mathbf{t} \in \mathbb{T}, r_n(\mathbf{t}) = \{u \in \mathbf{t}, |u| \leq n\}.$$

Classical results give that the distribution of a random tree τ on $\mathbb{T}$ is characterized by the family of probabilities $(\mathbb{P}(r_n(\tau) = \mathbf{t}), n \in \mathbb{N}^*, \mathbf{t} \in \mathbb{T}^{(n)})$.

2.2. Galton-Watson trees. Let $q = (q_n, n \in \mathbb{N})$ be a probability distribution on the nonnegative integers. We set $\mu = \sum_{n=0}^{+\infty} nq_n$ its mean and always suppose that $\mu < +\infty$.

A $\mathbb{T}_\partial$ -valued random tree τ is said to be Galton-Watson tree with offspring distribution q under $\mathbb{P}$ if, for every $h \in \mathbb{N}^*$ and every $\mathbf{t} \in \mathbb{T}_\partial^{(h)}$,

$$\mathbb{P}(r_h(\tau) = \mathbf{t}) = \prod_{u \in r_{h-1}(\mathbf{t})} q_{k_u(\mathbf{t})}.$$

The generation-size process defined by $(Z_n = z_n(\tau), n \in \mathbb{N})$ is the classical Galton-Watson process with offspring distribution q starting with a single individual at time 0.

As we will later consider inhomogeneous Galton-Watson trees (whose offspring distribution depends on the height of the node), we define for every $k \in \mathbb{N}$ the distribution $\mathbb{P}_k$ under which the generation-size process is a Galton-Watson process starting with a single individual at time k :

$$\forall h > k, \forall \mathbf{t} \in \mathbb{T}_k^{(h)}, \mathbb{P}_k(r_h(\tau) = \mathbf{t}) = \prod_{u \in r_{h-1}(\mathbf{t})} q_{k_u(\mathbf{t})}.$$

In other word, the random tree τ under $\mathbb{P}_k$ is distributed as $\mathbf{1}_k\tau$ under $\mathbb{P}$, and $\mathbb{P}$ is equal to $\mathbb{P}_\partial$.

Let f denote the generating function of q and for every $n \geq 1$, we set f_n the n -fold iterate of f :

$$f_1 = f, \quad \forall n \geq 1, \quad f_{n+1} = f_n \circ f.$$

Then f_n is the generating function of the random variable Z_n under $\mathbb{P}$.

We recall now the classical result on the extinction probability of the Galton-Watson tree and introduce some notations. We denote by $\{H(\tau) < +\infty\} = \bigcup_{n \in \mathbb{N}} \{Z_n = 0\}$ the extinction event and denote by κ the extinction probability:

$$(4) \quad \kappa = \mathbb{P}(H(\tau) < +\infty).$$

Then, κ is the smallest non-negative root of $f(s) = s$. Moreover, we can prove that

$$(5) \quad \forall s \in [0, 1), \quad \lim_{n \rightarrow +\infty} f_n(s) = \kappa.$$

We recall the three following cases:

- The sub-critical case ($\mu < 1$): $\kappa = 1$.
- The critical case ($\mu = 1$): $\kappa = 1$ (unless $q_1 = 1$ and then $\kappa = 0$).
- The super-critical case ($\mu > 1$): $\kappa \in [0, 1)$, the process has a positive probability of non-extinction.

In the super-critical case, we recall that

$$(6) \quad \mathfrak{a} = \min\{k \geq 0, \quad q_k > 0\}$$

and we say that we are in the Schroeder case if $\mathfrak{a} \leq 1$ (which implies $f'(\kappa) > 0$) and in the Bötcher case if $\mathfrak{a} > 1$ (in that case, we have $\kappa = f'(\kappa) = 0$).

It is easy to check that the process $(Z_n/\mu^n, n \in \mathbb{N})$ is a nonnegative martingale under $\mathbb{P}$ and hence converges a.s. toward a random variable denoted by W . Moreover, following [18], we know that, in the super-critical case, if q satisfies the so-called $L \log L$ condition i.e. $\mathbb{E}[Z_1 \log Z_1] < +\infty$, then W is non-degenerate and $\mathbb{P}(W = 0) = \kappa$.

Let us denote by ϕ the Laplace transform of W . Then, ϕ is the unique (up to a linear change of variable) solution of Schroeder's equation (see [18], Theorem 4.1):

$$(7) \quad \forall a \geq 0, \quad f(\phi(a)) = \phi(a\mu).$$

3. STANDARD LIMITING MARTINGALES

In this section, we study the penalization function

$$(8) \quad \varphi_p(x) = H_p(x)s^x$$

for some fixed integer $p \in \mathbb{N}^*$ and some fixed $s \in [0, 1)$ (or $s = 1$ in the critical case).

3.1. A formula for the conditional expectation. Let n, m be non-negative integers. According to the branching property, conditionally on $\mathcal{F}_n$, we have

$$Z_{n+m} \stackrel{(d)}{=} \sum_{j=1}^{Z_n} Z_m^{(j)}$$

where the sequence $(Z^{(j)}, j \geq 1)$ are i.i.d. copies of Z . Therefore we deduce that, for every $s \in [0, 1)$, we have

$$(9) \quad \mathbb{E}[s^{Z_{m+n}} | \mathcal{F}_n] = \mathbb{E} \left[\prod_{j=1}^{Z_n} s^{Z_m^{(j)}} \middle| \mathcal{F}_n \right] = \prod_{j=1}^{Z_n} \mathbb{E} [s^{Z_m^{(j)}}] = f_m(s)^{Z_n}.$$

Let us denote by

$$(10) \quad S_{i,p} = \left\{ (n_1, \dots, n_i) \in (\mathbb{N}^*)^i, \sum_{k=1}^i n_k = p \right\}.$$

We have the following result:

Lemma 3.1. *Let $p \in \mathbb{N}^*$ and let q be an offspring distribution with a finite p -th moment. For every $n, m \in \mathbb{N}$ and every $s \in [0, 1)$, we have*

$$(11) \quad \mathbb{E} \left[H_p(Z_{n+m}) s^{Z_{n+m}-p} \mid \mathcal{F}_n \right] = \sum_{i=1}^p H_i(Z_n) f_m(s)^{Z_n-i} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_m^{(n_j)}(s)}{n_j!}.$$

Proof. First recall Faà di Bruno's formula:

$$(12) \quad \frac{d^p}{dx^p} f(g(x)) = p! \sum_{i=1}^p \frac{1}{i!} f^{(i)}(g(x)) \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{g^{(n_j)}(x)}{n_j!}.$$

Using (9), we get

$$\begin{aligned} \mathbb{E} [H_p(Z_{n+m}) s^{Z_{n+m}-p} \mid \mathcal{F}_n] &= \frac{1}{p!} \frac{d^p}{ds^p} (\mathbb{E}[s^{Z_{n+m}} \mid \mathcal{F}_n]) = \frac{1}{p!} \frac{d^p}{ds^p} (f_m(s)^{Z_n}) \\ &= \sum_{i=1}^p H_i(Z_n) f_m(s)^{Z_n-i} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_m^{(n_j)}(s)}{n_j!} \end{aligned}$$

□

3.2. The limiting martingale for $s \in [0, 1)$ in the non-critical case.

Lemma 3.2. *Let $p \in \mathbb{N}^*$ and q be a non-critical offspring distribution that satisfies the $L \log L$ condition. We suppose that we are in the Schroeder case if q is super-critical. Then, there exists a positive function C_p , such that for all $s \in [0, 1)$:*

$$(13) \quad f_n^{(p)}(s) \underset{n \rightarrow +\infty}{\sim} C_p(s) \gamma^n$$

where $\gamma = f'(\kappa) \in (0, 1)$.

Note that the $L \log L$ condition in the sub-critical case is needed to avoid $C_1 \equiv 0$ (see [4] pp. 38).

Proof. The case $p = 1$ is classical (with $C_1(s) = 1$) and can be found in [4] (pp. 38). The rest of the proof is a generalisation of the case $p = 2$ found in [10].

Assume that (13) is true for all $j \leq p - 1$. Using again Faà di Bruno's formula, we get:

$$\begin{aligned} \frac{f_{n+1}^{(p)}(s)}{f_{n+1}'(s)} &= \frac{p!}{f_{n+1}'(s)} \sum_{i=1}^p \frac{1}{i!} f^{(i)}(f_n(s)) \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_n^{(n_j)}(s)}{n_j!} \\ &= \frac{p!}{f_{n+1}'(s)} \sum_{i=2}^p \frac{1}{i!} f^{(i)}(f_n(s)) \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_n^{(n_j)}(s)}{n_j!} \\ &\quad + \frac{f'(f_n(s)) f_n^{(p)}(s)}{f_n'(s) f'(f_n(s))}. \end{aligned}$$

Therefore, we get

$$(14) \quad \frac{f_{n+1}^{(p)}(s)}{f_{n+1}'(s)} - \frac{f_n^{(p)}(s)}{f_n'(s)} = \sum_{i=2}^p \frac{p!}{i!} f^{(i)}(f_n(s)) \frac{1}{f_{n+1}'(s)} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_n^{(n_j)}(s)}{n_j!}$$

For every $2 \leq i \leq p$ and every $(n_1, \dots, n_i) \in S_{i,p}$, as $n_j < p$ for every $j \leq i$, we can use the induction hypothesis and deduce that there exists a positive constant K such that

$$(15) \quad \frac{1}{f_{n+1}'(s)} \prod_{j=1}^i \frac{f_n^{(n_j)}(s)}{n_j!} \underset{n \rightarrow +\infty}{\sim} K \gamma^{n(i-1)}.$$

The continuity of $f^{(j)}$ and (5) imply that $\lim_{n \rightarrow \infty} f^{(j)}(f_n(s)) = f^{(j)}(\kappa)$ for all $j \geq 1$, thus formulas (14) and (15) implies that

$$\frac{f_{n+1}^{(p)}(s)}{f_{n+1}'(s)} - \frac{f_n^{(p)}(s)}{f_n'(s)} \underset{n \rightarrow +\infty}{\sim} K' \gamma^n$$

for some constant $K' > 0$.

As $\gamma \in (0, 1)$, uniformly on any compact of $(0, 1)$:

$$0 < C_p(s) := \lim_{n \rightarrow +\infty} \frac{f_n^{(p)}(s)}{f_n'(s)} = \frac{f_1^{(p)}(s)}{f_1'(s)} + \sum_{n \geq 1} \frac{f_{n+1}^{(p)}(s)}{f_{n+1}'(s)} - \frac{f_n^{(p)}(s)}{f_n'(s)} < +\infty$$

which is equivalent to $f_n^{(p)}(s) \underset{n \rightarrow +\infty}{\sim} C_p(s) f_n'(s)$. Applying again the lemma for $p = 1$ gives the result. $\square$

We can now state the main results concerning the limit of (2) with the penalization function (8). We must separate two cases for super-critical offspring distributions depending on $q_0 > 0$ (which is equivalent to $\kappa > 0$) or $q_0 = 0$ (which is equivalent to $\kappa = 0$).

Theorem 3.3. *Let $p \in \mathbb{N}$ and let q be a non-critical offspring distribution that admits a moment of order p (and satisfies the $L \log L$ condition if $p = 1$). We assume furthermore that $q_0 > 0$ (which is true if q is sub-critical). Then, for every $s \in [0, 1)$, every $n \in \mathbb{N}$ and every $\Lambda_n \in \mathcal{F}_n$, we have*

$$(16) \quad \lim_{m \rightarrow +\infty} \frac{\mathbb{E}[H_p(Z_{m+n}) s^{Z_{m+n}} \mathbf{1}_{\Lambda_n}]}{\mathbb{E}[H_p(Z_{m+n}) s^{Z_{m+n}}]} = \mathbb{E}[\widetilde{M}_n^{(p)} \mathbf{1}_{\Lambda_n}]$$

$$\text{with } \widetilde{M}_n^{(p)} = \begin{cases} \kappa^{Z_n-1} & \text{if } p = 0, \\ \frac{Z_n \kappa^{Z_n-1}}{f'(\kappa)^n} & \text{if } p \geq 1. \end{cases}$$

Proof. Let us first consider the case $p = 0$. Using Equation (9), we get

$$\frac{\mathbb{E}[s^{Z_{m+n}} \mid \mathcal{F}_n]}{\mathbb{E}[s^{Z_{m+n}}]} = \frac{f_m(s)^{Z_n}}{f_{m+n}(s)}.$$

As, for every $s \in [0, 1)$, $\lim_{m \rightarrow +\infty} f_m(s) = \kappa > 0$, we get

$$\lim_{m \rightarrow +\infty} \frac{\mathbb{E}[s^{Z_{m+n}} \mid \mathcal{F}_n]}{\mathbb{E}[s^{Z_{m+n}}]} = \kappa^{Z_n-1}.$$

Moreover, using the increasing property of $(f_n(s))_{n \geq 1}$ in s and n , we have

$$0 \leq \frac{f_m(s)^{Z_n}}{f_{m+n}(s)} \leq \frac{1}{q_0}$$

so the dominated convergence theorem gives the result.

Let us now suppose that $p \geq 1$. Applying (13), as $0 < \gamma < 1$, we get that, for every $i \geq 1$,

$$\sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_m^{(n_j)}(s)}{n_j!} = C\gamma^{mi} + o(\gamma^{mi})$$

for some constant C . Therefore, in (11), we get that the term for $i = 1$ is dominant when $m \rightarrow +\infty$ and therefore

$$\mathbb{E} \left[H_p(Z_{n+m}) s^{Z_{n+m}-p} \mid \mathcal{F}_n \right] = \frac{1}{p!} Z_n f_m(s)^{Z_n-1} f_m^{(p)}(s) + o(\gamma^m) = \frac{1}{p!} C_p(s) Z_n \kappa^{Z_n-1} \gamma^m + o(\gamma^m).$$

Moreover, as for every $1 \leq i \leq p$, we have

$$H_i(Z_n) f_m(s)^{Z_n-i} = H_i(Z_n) f_m(s)^{Z_n-i} \mathbb{1}_{Z_n \geq i} \leq H_i(Z_n),$$

we have by dominated convergence

$$\lim_{m \rightarrow +\infty} \mathbb{E} [H_i(Z_n) f_m(s)^{Z_n-i}] = \mathbb{E} [H_i(Z_n) \kappa^{Z_n-i}],$$

and by the same arguments as above, we get

$$\forall \Lambda_n \in \mathcal{F}_n, \quad \mathbb{E} [H_p(Z_{n+m}) s^{Z_{n+m}-p} \mathbb{1}_{\Lambda_n}] = \frac{1}{p!} C_p(s) \mathbb{E} [Z_n \kappa^{Z_n-1} \mathbb{1}_{\Lambda_n}] \gamma^m + o(\gamma^m).$$

Using (11) with $n = 0$ and (13), we have for $m \rightarrow +\infty$,

$$\mathbb{E} [H_p(Z_{m+n}) s^{Z_{m+n}-p}] = \frac{1}{p!} f_{n+m}^{(p)}(s) = \frac{1}{p!} C_p(s) \gamma^{n+m} + o(\gamma^{n+m}).$$

Combining these two asymptotics yields

$$\lim_{m \rightarrow +\infty} \frac{\mathbb{E} [H_p(Z_{m+n}) s^{Z_{m+n}} \mathbb{1}_{\Lambda_n}]}{\mathbb{E} [H_p(Z_{m+n}) s^{Z_{m+n}}]} = \mathbb{E} \left[\frac{Z_n \kappa^{Z_n-1}}{\gamma^n} \mathbb{1}_{\Lambda_n} \right] = \mathbb{E} \left[\frac{Z_n \kappa^{Z_n-1}}{f'(\kappa)^n} \mathbb{1}_{\Lambda_n} \right].$$

□

Remark 3.4. Let P be a polynomial of degree $p > 0$ that vanishes at 0. Therefore, there exists constants $(\alpha_k)_{1 \leq k \leq p}$ such that

$$P = \sum_{k=1}^p \alpha_k H_k.$$

Then, the previous asymptotics give, for every $n, m \in \mathbb{N}$ and every $\Lambda_n \in \mathcal{F}_n$,

$$\mathbb{E} [P(Z_{m+n}) s^{Z_{m+n}} \mathbb{1}_{\Lambda_n}] = \left(\sum_{k=1}^p \alpha_k \frac{s^k}{k!} C_k(s) \right) \mathbb{E} [Z_n \kappa^{Z_n-1} \mathbb{1}_{\Lambda_n}] \gamma^m + o(\gamma^m)$$

and

$$\mathbb{E} [P(Z_{m+n}) s^{Z_{m+n}}] = \left(\sum_{k=1}^p \alpha_k \frac{s^k}{k!} C_k(s) \right) \gamma^{m+n} + o(\gamma^m)$$

which implies that we obtain the same limit with P or with H_p in the penalizing function.

Recall Definition (6) of $\mathfrak{a}$.

Theorem 3.5. *Let q be a super-critical offspring distribution that admits a moment of order $p \in \mathbb{N}$, and let us suppose that $\mathfrak{a} > 0$ (or equivalently $q_0 = 0$). Then, for every $s \in (0, 1)$, every $n \in \mathbb{N}$ and every $\Lambda_n \in \mathcal{F}_n$, we have*

$$(17) \quad \lim_{m \rightarrow +\infty} \frac{\mathbb{E}[H_p(Z_{m+n})s^{Z_{m+n}}\mathbb{1}_{\Lambda_n}]}{\mathbb{E}[H_p(Z_{m+n})s^{Z_{m+n}}]} = \begin{cases} \mathbb{E}[q_1^{-n}\mathbb{1}_{Z_n=1}\mathbb{1}_{\Lambda_n}] & \text{if } \mathfrak{a} = 1, \\ \mathbb{E}\left[q_{\mathfrak{a}}^{-\frac{\mathfrak{a}^n-1}{\mathfrak{a}-1}}\mathbb{1}_{Z_n=\mathfrak{a}^n}\mathbb{1}_{\Lambda_n}\right] & \text{if } \mathfrak{a} \geq 2. \end{cases}$$

Proof. The reasoning is similar to the previous one.

Let us first consider the case $p = 0$, $\mathfrak{a} = 1$. In that case, we have (see [4] pp. 40 Corollary 1),

$$(18) \quad \forall s \in (0, 1), \quad f_m(s) \sim C_1(s)f'(0)^m = C_1(s)q_1^m$$

for a positive function $C_1(s)$. Therefore

$$\frac{\mathbb{E}[s^{Z_{m+n}} \mid \mathcal{F}_n]}{\mathbb{E}[s^{Z_{m+n}}]} = \frac{f_m(s)^{Z_n}}{f_{m+n}(s)} \sim q_1^{m(Z_n-1)} q_1^{-n}$$

which converges to 0 if $Z_n > 1$ and to q_1^{-n} if $Z_n = 1$. We conclude then by dominated convergence as $f_m(s)^{Z_n}/f_{m+n}(s) \leq f_2(0)^{-1}$ for all $m \geq 2$.

Let us now suppose that $p \geq 1$ and $\mathfrak{a} = 1$. Using (11), Lemma 3.2 and (18), we have

$$\mathbb{E}\left[H_p(Z_{n+m})s^{Z_{n+m}-p} \mid \mathcal{F}_n\right] = \sum_{i=1}^n C_i H_i(Z_n) q_1^{mZ_n} (1 + o(1))$$

for some constants C_i (note that $\gamma = f'(0) = q_1$ here) and

$$\mathbb{E}\left[H_p(Z_{n+m})s^{Z_{n+m}-p}\right] = \frac{1}{p!} f_{n+m}^{(p)}(s) = \frac{1}{p!} C_p(s) q_1^{n+m} + o(q_1^m)$$

which yields for some constant $K > 0$

$$\frac{\mathbb{E}\left[H_p(Z_{n+m})s^{Z_{n+m}} \mid \mathcal{F}_n\right]}{\mathbb{E}\left[H_p(Z_{n+m})s^{Z_{n+m}}\right]} \sim K q_1^{-n} q_1^{m(Z_n-1)} \sum_{i=1}^n C_i H_i(Z_n).$$

This ratio tends to 0 if $Z_n > 1$ and to $K' q_1^{-n}$ otherwise, with $K' > 0$. Dominated convergence Theorem ensures the existence of the limit (17) and we can easily find that $K' = 1$ recalling that necessarily $(K' q_1^{-n} \mathbb{1}_{Z_n=1})_{n \geq 0}$ is a martingale with mean equals to 1.

For the case $\mathfrak{a} \geq 2$, we use the asymptotics given in the following lemma whose proof is postponed after the current proof.

Lemma 3.6. *For every $p \in \mathbb{N}$ and every $s \in (0, 1)$, there exists a positive constant $K_p(s)$ such that*

$$f_m^{(p)}(s) = K_p(s) \mathfrak{a}^{mp} e^{\mathfrak{a}^m b(s)} (1 + o(1))$$

where

$$b(s) = \log s + \sum_{j=0}^{+\infty} \mathfrak{a}^{-j-1} \log \frac{f_{j+1}(s)}{f_j(s)^{\mathfrak{a}}},$$

In that case, we have for $p = 0$ as $m \rightarrow +\infty$

$$\frac{\mathbb{E}[s^{Z_{m+n}} \mid \mathcal{F}_n]}{\mathbb{E}[s^{Z_{m+n}}]} = \frac{f_m(s)^{Z_n}}{f_{m+n}(s)} \sim K_0(s)^{Z_n-1} e^{\mathfrak{a}^m b(s)(Z_n-\mathfrak{a}^n)} \xrightarrow{m \rightarrow +\infty} K_0(s)^{\mathfrak{a}^n-1} \mathbb{1}_{Z_n=\mathfrak{a}^n}$$

since $b(s) < 0$. We conclude either by saying that $K_0(s) = q_a^{-1/a-1}$ by [7] Lemma 10, or by using the fact that the limit is a martingale with mean 1.

For $p \geq 1$, we use Lemma 3.6 to get that, for every $1 \leq i \leq p$ and every $(n_1, \dots, n_i) \in S_{i,p}$, we have as $m \rightarrow +\infty$,

$$\prod_{j=1}^i \frac{f_m^{(n_j)}(s)}{n_j!} \sim \prod_{j=1}^i K_{n_j}(s) a^{m \cdot n_j} e^{a^m b(s)} / n_j! = K(s) a^{mp} e^{a^m b(s)i}$$

for some constant $K(s)$. Hence, we have for $1 \leq i \leq p$,

$$\sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_m^{(n_j)}(s)}{n_j!} \sim \tilde{K}_i(s) a^{mp} e^{a^m b(s)i}$$

for another constant $\tilde{K}_i(s)$ since all the terms in the sum are nonnegative and of the same order.

Finally, using (11), we get

$$\begin{aligned} \mathbb{E} [H_p(Z_{n+m}) s^{Z_{n+m}-p} \mid \mathcal{F}_n] &= \sum_{i=1}^p H_i(Z_n) f_m(s)^{Z_n-i} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_m^{(n_j)}(s)}{n_j!} \\ &\sim \sum_{i=1}^p K_0(s)^{Z_n-i} \tilde{K}_i(s) H_i(Z_n) e^{a^m (Z_n-i)b(s)} a^{mp} e^{a^m b(s)i} \\ &= \hat{K}(s, Z_n) e^{a^m Z_n b(s)} a^{mp} \end{aligned}$$

for some function $\hat{K}$, again since all the terms of the sum are nonnegative and of the same order.

This gives

$$\frac{\mathbb{E} [H_p(Z_{n+m}) s^{Z_{n+m}-p} \mid \mathcal{F}_n]}{\mathbb{E} [H_p(Z_{n+m}) s^{Z_{n+m}-p}]} \sim \frac{\hat{K}(s, Z_n)}{K_p(s)} a^{-np} e^{a^m b(s)(Z_n-a^n)} \xrightarrow{m \rightarrow +\infty} C_n \mathbb{1}_{Z_n=a^n}$$

where C_n is a constant depending on n that is computed again by saying that the limit is a martingale with mean 1. $\square$

Remark 3.7. The same arguments as in Remark 3.4 can be used to show that the limit does not depend of the choice of the polynomial

We now finish this section with the proof of Lemma 3.6.

Proof of Lemma 3.6. In the proof, the letter K will denote a constant that depends on s and may change from line to line.

Lemma 10 in [7] states that, for every $\delta > 0$ and every $s \in (0, 1 - \delta)$, we have

$$f_m(s) = q_a^{-1/(a-1)} e^{a^m b(s)} \left(1 + o(e^{-\delta a^m})\right)$$

which implies the result for $p = 0$.

To prove the result for $p = 1$, we follow the same ideas as in the proof of [7], Lemma 10. We still consider $s \in (0, 1 - \delta)$ for some $\delta > 0$. First, we have

$$f'_{m+1}(s) = f'_m(s) f'(f_m(s)) = f'_m(s) \sum_{k=0}^{+\infty} (a+k) q_{a+k} f_m(s)^{a+k-1},$$

which gives

$$0 \leq \frac{f'_{m+1}(s)}{f'_m(s) \mathfrak{a} q_{\mathfrak{a}} f_m(s)^{\mathfrak{a}-1}} - 1 \leq \sum_{k=1}^{+\infty} \frac{(\mathfrak{a} + k) q_{\mathfrak{a}+k}}{\mathfrak{a} q_{\mathfrak{a}}} f_m(s) \leq K e^{\mathfrak{a}^m b(s)}$$

by Lemma 13 of [7]. Therefore, as $\ln(1+u) \leq u$ for every nonnegative u , we have

$$0 \leq \ln \left(\frac{f'_{m+1}(s)}{f'_m(s) \mathfrak{a} q_{\mathfrak{a}} f_m(s)^{\mathfrak{a}-1}} \right) \leq K e^{\mathfrak{a}^m b(s)}$$

which implies that the series

$$\sum_{n=0}^{+\infty} \ln \left(\frac{f'_{m+1}(s)}{f'_m(s) \mathfrak{a} q_{\mathfrak{a}} f_m(s)^{\mathfrak{a}-1}} \right)$$

converges. Using the asymptotics for $f_m(s)$ of Lemma 10 of [7] we get that

$$\ln \left(\frac{f'_{m+1}(s)}{f'_m(s) \mathfrak{a} e^{\mathfrak{a}^m (\mathfrak{a}-1) b(s)}} \right) \sim \ln \left(\frac{f'_{m+1}(s)}{f'_m(s) \mathfrak{a} q_{\mathfrak{a}} f_m(s)^{\mathfrak{a}-1}} \right)$$

and hence that the series

$$\tilde{b}(s) := \sum_{m=0}^{+\infty} \ln \left(\frac{f'_{m+1}(s)}{f'_m(s) \mathfrak{a} e^{\mathfrak{a}^m (\mathfrak{a}-1) b(s)}} \right)$$

converges.

Moreover, as

$$\ln \frac{f'_m(s)}{\mathfrak{a}^m e^{(\mathfrak{a}^m-1)b(s)}} = \tilde{b}(s) - \sum_{k=m}^{+\infty} \ln \left(\frac{f'_{k+1}(s)}{f'_k(s) \mathfrak{a} e^{\mathfrak{a}^k (\mathfrak{a}-1) b(s)}} \right) = \tilde{b}(s) + o(1),$$

we obtain

$$\frac{f'_m(s)}{\mathfrak{a}^m e^{(\mathfrak{a}^m-1)b(s)}} = e^{\tilde{b}(s)} (1 + o(1))$$

which is the looked after formula for $p = 1$.

We finish the proof by induction on p as for the proof of Lemma 3.2. Let $p \geq 2$ and let us suppose that the asymptotics of Lemma 3.6 are true for every $j < p$. Recall Equation (14)

$$(19) \quad \frac{f_{m+1}^{(p)}(s)}{f_{m+1}'(s)} - \frac{f_m^{(p)}(s)}{f_m'(s)} = \sum_{i=2}^p \frac{p!}{i!} f^{(i)}(f_m(s)) \frac{1}{f_{m+1}'(s)} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_m^{(n_j)}(s)}{n_j!}$$

By the induction assumption, we have for every $1 \leq i \leq p$, using the same computations as in the proof of Theorem 3.5,

$$\sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_m^{(n_j)}(s)}{n_j!} \sim K \mathfrak{a}^{mp} e^{\mathfrak{a}^m b(s) i}.$$

We also have

$$f'_{m+1}(s) \sim K \mathfrak{a}^m e^{\mathfrak{a}^{m+1} b(s)}$$

and for every $i \geq 2$,

$$\begin{aligned} f^{(i)}(f_m(s)) &= \begin{cases} K f_m(s)^{\mathfrak{a}-i}(1+o(1)) & \text{if } i < \mathfrak{a}, \\ O(1) & \text{if } i \geq \mathfrak{a}, \end{cases} \\ &= \begin{cases} K e^{\mathfrak{a}^m(\mathfrak{a}-i)b(s)}(1+o(1)) & \text{if } i < \mathfrak{a}, \\ O(1) & \text{if } i \geq \mathfrak{a}. \end{cases} \end{aligned}$$

Hence, in the sum of (19), the terms for $i \leq \mathfrak{a}$ (which exist since $\mathfrak{a} \geq 2$) are dominant and of order $K \mathfrak{a}^{m(p-1)}$.

We get

$$\frac{f_{m+1}^{(p)}(s)}{f_{m+1}'(s)} - \frac{f_m^{(p)}(s)}{f_m'(s)} \sim K \mathfrak{a}^{m(p-1)}$$

and, as the series diverge ($\mathfrak{a}^{p-1} > 1$), the partial sums are also equivalent, which gives

$$f_m^{(p)}(s) \sim K \mathfrak{a}^{m(p-1)} f_m'(s) \sim K \mathfrak{a}^{mp} e^{\mathfrak{a}^m b(s)}$$

using the result for $p = 1$. □

3.3. The limiting martingale for $s \in [0, 1]$ in the critical case. We finish with the result for a critical offspring distribution. As the arguments are the same as for the proof of Theorem 3.3, we only give the main lines in the proof of the following theorem.

Theorem 3.8. *Let q be a critical offspring distribution that admits a moment of order $p \in \mathbb{N}$. Then, for every $s \in [0, 1]$, every $n \in \mathbb{N}$ and every $\Lambda_n \in \mathcal{F}_n$, we have*

$$\lim_{m \rightarrow +\infty} \frac{\mathbb{E}[H_p(Z_{m+n}) s^{Z_{m+n}} \mathbf{1}_{\Lambda_n}]}{\mathbb{E}[H_p(Z_{m+n}) s^{Z_{m+n}}]} = \begin{cases} \mathbb{E}[\mathbf{1}_{\Lambda_n}] & \text{if } p = 0, \\ \mathbb{E}[Z_n \mathbf{1}_{\Lambda_n}] & \text{if } p \geq 1. \end{cases}$$

Proof. We first study the case $s \in [0, 1)$.

For $p = 0$, the proof of Theorem 3.3 still applies with $\kappa = 1$.

For $p = 1$, note that according to the dominated convergence theorem

$$\begin{aligned} \frac{\mathbb{E}[Z_{n+m} s^{Z_{n+m}-1} | \mathcal{F}_n]}{f_{m+n}'(s)} &= \frac{\mathbb{E}[Z_{n+m} s^{Z_{n+m}-1} | \mathcal{F}_n]}{\mathbb{E}[Z_{n+m} s^{Z_{n+m}-1}]} = \frac{Z_n f_m(s)^{Z_n-1} f_m'(s)}{\mathbb{E}[Z_n f_m(s)^{Z_n-1} f_m'(s)]} = \frac{Z_n f_m(s)^{Z_n-1}}{\mathbb{E}[Z_n f_m(s)^{Z_n-1}]} \\ &\xrightarrow{m \rightarrow +\infty} \frac{Z_n}{\mathbb{E}[Z_n]} = Z_n \end{aligned}$$

giving our result. Moreover we can deduce from this limit's ratio that for all $n \geq 0$, when m goes to infinity

$$(20) \quad f_{m+n}'(s) \sim f_m'(s).$$

We then replace Lemma 3.2 by the following asymptotics for $f_n^{(p)}(s)$ whose proof is postponed at the end of the section.

Lemma 3.9. *In the critical case, for every $p \geq 1$, there exists a positive function C_p , such that for all $s \in [0, 1]$:*

$$(21) \quad f_n^{(p)}(s) \underset{n \rightarrow +\infty}{\sim} C_p(s) f_n'(s)$$

The result then follows using the same arguments as in the proof of Theorem 3.3.

Let us now consider the case $s = 1$. The case $p = 0$ is trivial, so let us suppose that $p \geq 1$. Equation (11) applied to $s = 1$ and $m = 1$ gives

$$\mathbb{E}[H_p(Z_{n+1})|\mathcal{F}_n] = \sum_{i=1}^p H_i(Z_n) \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{1}{n_j!} \mathbb{E}[H_{n_j}(Z_1)]$$

and an easy induction on n and p gives the following lemma.

Lemma 3.10. *Let q be a critical offspring distribution that admits a moment of order p . Then there exists a polynomial P of degree $p - 1$ such that, for every $n \geq 0$,*

$$\mathbb{E}[H_p(Z_n)] = P(n).$$

This gives asymptotics of $\mathbb{E}[H_p(Z_n)]$ of the form cn^{p-1} as $n \rightarrow +\infty$. Plugging these asymptotics in (11) and arguing as in the proof of Theorem 3.3 gives the result. $\square$

Proof of Lemma 3.9. We first need to prove that $\sum_{n \geq 0} f'_n(s) < +\infty$. Let G be the function defined on $[0, 1)$ by

$$(22) \quad G(s) := \sum_{k \geq 1} s^k \sum_{n \geq 0} \mathbb{P}(Z_n = k) = \sum_{n \geq 0} \sum_{k \geq 1} s^k \mathbb{P}(Z_n = k) = \sum_{n \geq 0} (f_n(s) - f_n(0)).$$

According to [12] pp.584, there exists a function U , such that for $s \in [0, 1)$:

$$\lim_{n \rightarrow +\infty} n^2(f_n(s) - f_n(0)) = U(s) < \infty$$

implying that G is a power series that converges on $[0, 1)$ and we have on this interval

$$G'(s) = \sum_{n \geq 0} f'_n(s) < +\infty.$$

The rest of the proof is very similar to the one of Lemma 3.2: using (20) and the induction hypothesis, the equivalent of formula (15) is

$$(23) \quad \frac{1}{f'_{n+1}(s)} \prod_{j=1}^i \frac{f_n^{(n_j)}(s)}{n_j!} \underset{n \rightarrow +\infty}{\sim} K(f'_n(s))^{(i-1)}$$

implying that

$$\frac{f_{n+1}^{(p)}(s)}{f'_{n+1}(s)} - \frac{f_n^{(p)}(s)}{f'_n(s)} \underset{n \rightarrow +\infty}{\sim} K' f'_n(s)$$

for some constant $K' > 0$. Consequently

$$0 < C_p(s) := \lim_{n \rightarrow +\infty} \frac{f_n^{(p)}(s)}{f'_n(s)} = \frac{f_1^{(p)}(s)}{f'_1(s)} + \sum_{n \geq 1} \frac{f_{n+1}^{(p)}(s)}{f'_{n+1}(s)} - \frac{f_n^{(p)}(s)}{f'_n(s)} < +\infty$$

which is equivalent to $f_n^{(p)}(s) \underset{n \rightarrow +\infty}{\sim} C_p(s) f'_n(s)$. $\square$

4. A NEW MARTINGALE IN THE SUPER-CRITICAL CASE WHEN $s \rightarrow 1$

We are now considering the same penalization function (8) but with $s = 1$ or with s replaced by a sequence (s_n) that tends to 1. More precisely, we consider functions of the form

$$(24) \quad \varphi_p(n, x) = H_p(x) e^{-ax/\mu^n}$$

for some non-negative constant a .

4.1. The limiting martingale. Recall that H_p denotes the p -th Hilbert polynomial defined by (3) and ϕ the Laplace transform of W the limit of the martingale $(Z_n/\mu^n)_{n \geq 0}$.

For every $a \geq 0$, every $p \in \mathbb{N}^*$ and every $n \in \mathbb{N}$, we set, for every $x \in \mathbb{R}$,

$$(25) \quad G_n^{(p)}(x) = \frac{p!}{\phi^{(p)}(a)} \sum_{i=1}^p a_i^{(p)}(n) \phi(a/\mu^n)^{x-i} H_i(x)$$

with

$$(26) \quad a_i^{(p)}(n) = \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{r=1}^i \frac{\phi^{(n_r)}(a/\mu^n)}{n_r!}.$$

Let us first state the following relation between the coefficients $a_i^{(p)}(n)$ that will be used further.

Lemma 4.1. *For every $i \in \mathbb{N}^*$ and every $(s_1, \dots, s_i) \in (\mathbb{N}^*)^i$, let us set $w = \sum_{j=1}^i s_j$. Then, we have for every $n, p \geq 0$,*

$$\sum_{(\ell_1, \dots, \ell_i) \in S_{i,p}} \prod_{j=1}^i a_{s_j}^{(\ell_j)}(n) = a_w^{(p)}(n)$$

Proof. Let us consider the polynomial

$$P(X) = \sum_{k=1}^p \frac{\phi^{(k)}(a/\mu^n)}{k!} X^k.$$

Then, by (26), $a_s^{(\ell)}(n)$ is the coefficient of order ℓ of the polynomial P^s for every $s \leq \ell \leq p$. The lemma is then just a consequence of the formula

$$\prod_{j=1}^i P^{s_j}(X) = P^w(X).$$

□

Theorem 4.2. *Let $p \in \mathbb{N}^*$. Let q be a super-critical offspring distribution that admits a moment of order p . Then, for every $a \geq 0$, every $n \in \mathbb{N}$, and every $\Lambda_n \in \mathcal{F}_n$,*

$$\lim_{m \rightarrow +\infty} \frac{\mathbb{E} \left[H_p(Z_{m+n}) e^{-aZ_{m+n}/\mu^{m+n}} \mathbf{1}_{\Lambda_n} \right]}{\mathbb{E} \left[H_p(Z_{m+n}) e^{-aZ_{m+n}/\mu^{m+n}} \right]} = \mathbb{E} \left[\frac{1}{\mu^{pn}} G_n^{(p)}(Z_n) \mathbf{1}_{\Lambda_n} \right].$$

Proof. Let us first remark that for all $k \in \{0, \dots, m\}$

$$(27) \quad \frac{1}{\mu^{mk}} f_m^{(k)} \left(e^{-\frac{a}{\mu^{m+n}}} \right) = \mathbb{E} \left[e^{-a \frac{Z_m - k}{\mu^{n+m}}} \prod_{i=1}^k \frac{Z_m - i + 1}{\mu^m} \right] \xrightarrow{m \rightarrow +\infty} (-1)^k \phi^{(k)} \left(\frac{a}{\mu^n} \right).$$

And, by the same argument, we have

$$\frac{1}{\mu^{p(m+n)}} \mathbb{E} \left[H_p(Z_{m+n}) e^{-a \frac{Z_{m+n}}{\mu^{m+n}}} \right] = \frac{1}{p!} \mathbb{E} \left[e^{-a \frac{Z_{m+n}}{\mu^{m+n}}} \prod_{i=1}^p \frac{Z_{m+n} - i + 1}{\mu^{m+n}} \right] \xrightarrow{m \rightarrow +\infty} (-1)^p \frac{\phi^{(p)}(a)}{p!}.$$

Using Lemma 3.1 and (27), we get

$$\begin{aligned} & \frac{1}{\mu^{pm}} \mathbb{E} \left[H_p(Z_{m+n}) e^{-a \frac{Z_{m+n}}{\mu^{m+n}}} \mid \mathcal{F}_n \right] \\ &= e^{-\frac{pa}{\mu^{m+n}}} \sum_{i=1}^p H_i(Z_n) f_m(e^{-\frac{a}{\mu^{m+n}}})^{Z_n-i} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{f_m^{(n_j)}(e^{-\frac{a}{\mu^{m+n}}})}{n_j! \mu^{n_j m}} \\ &\xrightarrow{m \rightarrow +\infty} \sum_{i=1}^p H_i(Z_n) \phi(a/\mu^n)^{Z_n-i} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i (-1)^{n_j} \frac{\phi^{(n_j)}(a/\mu^n)}{n_j!} \\ &= (-1)^p \sum_{i=1}^p H_i(Z_n) \phi(a/\mu^n)^{Z_n-i} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{\phi^{(n_j)}(a/\mu^n)}{n_j!}. \end{aligned}$$

Again, for every $1 \leq i \leq p$, we have $H_i(Z_n) f_m(e^{-a/\mu^{m+n}})^{Z_n-i} \leq H_i(Z_n)$ so, by dominated convergence, we get

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{\mathbb{E} \left[H_p(Z_{m+n}) e^{-a \frac{Z_{m+n}}{\mu^{m+n}}} \mathbf{1}_{\Lambda_n} \right]}{\mathbb{E} \left[H_p(Z_{m+n}) e^{-a \frac{Z_{m+n}}{\mu^{m+n}}} \right]} \\ &= \frac{1}{\mu^{pn}} \frac{p!}{\phi^{(p)}(a)} \sum_{i=1}^p H_i(Z_n) \phi(a/\mu^n)^{Z_n-i} \sum_{(n_1, \dots, n_i) \in S_{i,p}} \prod_{j=1}^i \frac{\phi^{(n_j)}(a/\mu^n)}{n_j!}. \end{aligned}$$

□

We end this subsection with the following uniqueness result concerning the limiting martingale in the homogeneous case i.e. $a = 0$.

Proposition 4.3. *Let $p \geq 1$. There exists a unique polynomial P_p of degree p that vanishes at 0 such that the process $(X_n^{(p)})_{n \geq 0}$ defined by*

$$X_n^{(p)} = \frac{1}{\mu^{pn}} P_p(Z_n)$$

is a martingale with mean 1.

Proof. Existence is given by Theorem 4.2.

For uniqueness, let us write

$$P_k = \sum_{i=1}^k c_i^{(k)} H_i(Z_n)$$

and let us suppose that $X_n^{(k)}$ is a martingale with mean 1 for every $1 \leq k \leq p$. This implies by taking the expectation that, for every $n \geq 0$ and every $k \leq p$,

$$(28) \quad \sum_{i=1}^k \frac{1}{i!} c_i^{(k)} f_n^{(i)}(1) = \mu^{kn}.$$

If we set for $1 \leq i, j \leq p$

$$f_{ij} = f_{i-1}^{(j)}(1), \quad c_{ij} = \begin{cases} c_i^{(j)} & \text{if } i \leq j, \\ 0 & \text{if } i > j, \end{cases} \quad m_{ij} = \mu^{(i-1)j}$$

and if we consider the square matrices of order p

$$F = (f_{ij})_{1 \leq i, j \leq p}, \quad C = (c_{ij})_{1 \leq i, j \leq p}, \quad M = (m_{ij})_{1 \leq i, j \leq p},$$

Equations (28) for $0 \leq n \leq p-1$ write

$$(29) \quad FC = M$$

where C contains the unknown variables.

We know that, if $c_i^{(k)} = a_i^{(k)}$ where the $a_i^{(k)}$ are defined by (26) with $a = 0$ (and hence do not depend on n), C is indeed a solution of Equation (29) and is triangular with positive coefficients and hence $\det C \neq 0$. M is a Vandermonde matrix and hence also satisfies $\det M \neq 0$. Equation (29) hence implies $\det F = \det M / \det C \neq 0$ which proves that F is invertible and that (29) has a unique solution. $\square$

This proposition implies in particular that the choice of H_p in Theorem 4.2 (if $a = 0$) in the penalizing function is not relevant and any other polynomial of degree $p > 1$ that vanishes at 0 gives the same limit.

4.2. Distribution of the penalized tree. In this section, we fix an integer $p \geq 0$ and consider an offspring distribution q that admits a p -th moment (and that satisfies the $L \log L$ condition if $p < 2$).

For every $n \geq n_0$, we consider the function

$$G_{n,n_0}^{(p)}(x) = \begin{cases} \frac{\phi(a/\mu^n)^x}{\phi(a/\mu^{n_0})} & \text{if } p = 0 \\ \frac{p!}{\phi^{(p)}(a/\mu^{n_0})} \sum_{i=1}^p a_i^{(p)}(n) H_i(x) \phi(a/\mu^n)^{x-i} & \text{if } p \geq 1 \end{cases}$$

with $a_i^{(p)}(n)$ defined by (26) and we consider the martingale

$$M_{n,n_0}^{(p)} = \frac{1}{\mu^{p(n-n_0)}} G_{n,n_0}^{(p)}(Z_n)$$

We then define a new probability measure $\mathbb{Q}_{n_0}^p$ on $\mathbb{T}$ by

$$(30) \quad \forall n \geq n_0, \quad \frac{d\mathbb{Q}_{n_0}^{(p)}}{d\mathbb{P}_{n_0} \mid \mathcal{F}_n} = M_{n,n_0}^{(p)}.$$

We now define another probability measure $\mathbf{Q}_{n_0}^{(p)}$ on $\mathbb{T}$ as follows

Definition 4.4. Under $\mathbf{Q}_{n_0}^{(p)}$, the random tree τ is distributed as an inhomogeneous multi-type Galton-Watson tree as follows

- The types of the nodes run from 0 to p .
- The root of τ is of type p and starts at height n_0 .

- A node of type ℓ at height n gives, independently of the other nodes, k offspring with respective types $(\ell_1, \dots, \ell_k)$ such that $\ell_1 + \dots + \ell_k = \ell$ with probability

$$q_k \frac{1}{\mu^\ell} \frac{\ell!}{\phi^{(\ell)}(a/\mu^n)} \prod_{j=1}^k \frac{\phi^{(\ell_j)}(a/\mu^{n+1})}{\ell_j!}.$$

Remark 4.5. A node of type 0 at height n gives k offspring with probability

$$q_k^0(n) = q_k \frac{\phi(a/\mu^{n+1})^k}{\phi(a/\mu^n)},$$

all of them being of type 0.

Remark also that $q_k^0(n) = q_k$ if $a = 0$.

Remark 4.6. If a node is of type $\ell > 0$, the condition $\ell_1 + \dots + \ell_k = \ell$ implies that this node has at least one offspring with non-zero type.

Remark 4.7. The last property also writes: A node of type ℓ at height n gives, independently of the other nodes, k offspring, $k-i$ being of type 0, and i of respective types $(\ell_1, \dots, \ell_i) \in S_{i,\ell}$, with probability

$$(31) \quad q_k \frac{\ell!}{\mu^\ell} \frac{\phi(a/\mu^{n+1})^{k-i}}{\phi^{(\ell)}(a/\mu^n)} \binom{k}{i} \prod_{j=1}^i \frac{\phi^{(\ell_j)}(a/\mu^{n+1})}{\ell_j!}.$$

The i nodes with non-zero types are uniformly chosen among the k offspring.

This equivalent formulation will be used in all the next proofs.

Lemma 4.8. Equation (31) indeed defines a probability distribution.

Proof. We must prove that

$$\sum_{k=1}^{+\infty} \sum_{i=1}^{k \wedge \ell} \sum_{(\ell_1, \dots, \ell_i) \in S_{i,\ell}} q_k \frac{\ell!}{\mu^\ell} \frac{\phi(a/\mu^{n+1})^{k-i}}{\phi^{(\ell)}(a/\mu^n)} \binom{k}{i} \prod_{j=1}^i \frac{\phi^{(\ell_j)}(a/\mu^{n+1})}{\ell_j!} = 1.$$

First remark that formula (26) gives:

$$\sum_{(\ell_1, \dots, \ell_i) \in S_{i,\ell}} \prod_{j=1}^i \frac{\phi^{(\ell_j)}(a/\mu^{n+1})}{\ell_j!} = a_i^{(\ell)}(n+1).$$

Now, as $M_{n+1,n}^{(\ell)}$ is a martingale with mean one, we have by taking the expectation

$$\begin{aligned} \frac{\ell!}{\mu^\ell \phi^{(\ell)}(a/\mu^n)} \sum_{i=1}^{\ell} a_i^{(\ell)}(n+1) \mathbb{E}_n [H_i(Z_{n+1}) \phi(a/\mu^{n+1})^{Z_{n+1}-i}] &= 1 \\ \iff \frac{\ell!}{\mu^\ell \phi^{(\ell)}(a/\mu^n)} \sum_{i=1}^{\ell} a_i^{(\ell)}(n+1) \sum_{k=i}^{+\infty} q_k H_i(k) \phi(a/\mu^{n+1})^{k-i} &= 1, \end{aligned}$$

which ends the proof by inverting the sums and noting that $H_i(k) = \binom{k}{i}$. $\square$

Theorem 4.9. For every $n_0 \geq 0$ the probability measures $\mathbf{Q}_{n_0}^{(p)}$ and $\mathbf{Q}_{n_0}^{(p)}$ coincide.

Proof. To prove the theorem, it suffices to prove that,

$$(32) \quad \forall n \geq n_0, \forall \mathbf{t} \in \mathbb{T}_{n_0}^n, \mathbf{Q}_{n_0}^{(p)}(r_n(\tau) = \mathbf{t}) = \mathbb{Q}_{n_0}^{(p)}(r_n(\tau) = \mathbf{t})$$

We prove this formula by induction on p .

For $p = 0$, we have, for every $n > n_0$ (the case $n = n_0$ is trivial as the tree $r_n(\tau)$ is reduced to the root),

$$\begin{aligned} \mathbf{Q}_{n_0}^{(0)}(r_n(\tau) = \mathbf{t}) &= \prod_{r=n_0}^{n-1} \prod_{\{u \in \mathbf{t}, |u|=r\}} q_{k_u(\mathbf{t})}^0(r) = \prod_{r=n_0}^{n-1} \prod_{\{u \in \mathbf{t}, |u|=r\}} q_{k_u(\mathbf{t})} \frac{\phi(a/\mu^{r+1})^{k_u(\mathbf{t})}}{\phi(a/\mu^r)} \\ &= \left(\prod_{r=n_0}^{n-1} \frac{\phi(a/\mu^{r+1})^{z_{r+1}(\mathbf{t})}}{\phi(a/\mu^r)^{z_r(\mathbf{t})}} \right) \mathbb{P}_{n_0}(r_n(\tau) = \mathbf{t}) \\ &= \frac{\phi(a/\mu^n)^{z_n(\mathbf{t})}}{\phi(a/\mu^{n_0})} \mathbb{P}_{n_0}(r_n(\tau) = \mathbf{t}) = \mathbb{Q}_{n_0}^{(0)}(r_n(\tau) = \mathbf{t}) \end{aligned}$$

since $z_{n_0}(\mathbf{t}) = 1$.

Let us now suppose that (32) is true for every $p' < p$. We prove that the property is true at rank p by induction on n .

We have already mentioned that the formula is trivially true for $n = n_0$.

Let us now fix $n > n_0$ and let us suppose that the formula is true at rank p for every $n' < n$ and let us prove it for n . Let $\mathbf{t} \in \mathbb{T}_{n_0}^{(n)}$ and let us denote by k_0 the number of offspring of the root of $\mathbf{t}$. We denote by $\mathbf{t}_1, \dots, \mathbf{t}_{k_0}$ the (ordered) sub-trees of $\mathbf{t}$ above the first generation. By decomposing according to the offspring of the root, we have

$$\begin{aligned} \mathbf{Q}_{n_0}^{(p)}(r_n(\tau) = \mathbf{t}) &= \frac{p!}{\mu^p} q_{k_0} \sum_{i=1}^{k_0 \wedge p} \binom{k_0}{i} \frac{\phi(a/\mu^{n_0+1})^{k_0-i}}{\phi^{(p)}(a/\mu^{n_0})} \sum_{(\ell_1, \dots, \ell_i) \in S_{i,p}} \left(\prod_{j=1}^i \frac{\phi^{(\ell_j)}(a/\mu^{n_0+1})}{\ell_j!} \right) \\ &\quad \times \frac{1}{\binom{k_0}{i}} \sum_{1 \leq r_1 < \dots < r_i \leq k_0} \left(\prod_{j=1}^i \mathbf{Q}_{n_0+1}^{(\ell_j)}(r_n(\tau) = \mathbf{t}_{r_j}) \right) \times \left(\prod_{\substack{1 \leq k \leq k_0 \\ k \notin \{r_1, \dots, r_i\}}} \mathbf{Q}_{n_0+1}^{(0)}(r_n(\tau) = \mathbf{t}_k) \right). \end{aligned}$$

Therefore, as

$$\mathbb{P}_{n_0}(r_n(\tau) = \mathbf{t}) = q_{k_0} \prod_{j=1}^{k_0} \mathbb{P}_{n_0+1}(r_n(\tau) = \mathbf{t}_j),$$

we have

$$\begin{aligned}
\frac{\mu^{p(n-n_0)} \mathbf{Q}_{n_0}^{(p)}(r_n(\tau) = \mathbf{t})}{\mathbb{P}_{n_0}(r_n(\tau) = t)} &= p! \sum_{i=1}^{k_0 \wedge p} \frac{\phi(a/\mu^{n_0+1})^{k_0-i}}{\phi^{(p)}(a/\mu^{n_0})} \sum_{(\ell_1, \dots, \ell_i) \in S_{i,p}} \left(\prod_{j=1}^i \frac{\phi^{(\ell_j)}(a/\mu^{n_0+1})}{\ell_j!} \right) \\
&\quad \times \sum_{1 \leq r_1 < \dots < r_i \leq k_0} \left(\prod_{j=1}^i \frac{\mathbf{Q}_{n_0+1}^{(\ell_j)}(r_n(\tau) = \mathbf{t}_{r_j})}{\mathbb{P}_{n_0+1}(r_n(\tau) = \mathbf{t}_{\ell_j})} \mu^{\ell_j(n-n_0-1)} \right) \\
&\quad \times \left(\prod_{\substack{1 \leq k \leq k_0 \\ k \notin \{r_1, \dots, r_i\}}} \frac{\mathbf{Q}_{n_0+1}^{(0)}(r_n(\tau) = \mathbf{t}_k)}{\mathbb{P}_{n_0+1}(r_n(\tau) = \mathbf{t}_k)} \right) \\
&= p! \sum_{i=1}^{k_0 \wedge p} \frac{\phi(a/\mu^{n_0+1})^{k_0-i}}{\phi^{(p)}(a/\mu^{n_0})} \sum_{(\ell_1, \dots, \ell_i) \in S_{i,p}} \left(\prod_{j=1}^i \frac{\phi^{(\ell_j)}(a/\mu^{n_0+1})}{\ell_j!} \right) \\
&\quad \times \sum_{1 \leq r_1 < \dots < r_i \leq k_0} \left(\prod_{j=1}^i \frac{\mathbb{Q}_{n_0+1}^{(\ell_j)}(r_n(\tau) = \mathbf{t}_{r_j})}{\mathbb{P}_{n_0+1}(r_n(\tau) = \mathbf{t}_{\ell_j})} \mu^{\ell_j(n-n_0-1)} \right) \\
&\quad \times \left(\prod_{\substack{1 \leq k \leq k_0 \\ k \notin \{r_1, \dots, r_i\}}} \frac{\mathbb{Q}_{n_0+1}^{(0)}(r_n(\tau) = \mathbf{t}_k)}{\mathbb{P}_{n_0+1}(r_n(\tau) = \mathbf{t}_k)} \right).
\end{aligned}$$

by the induction assumption on p for $\ell_j < p$ (i.e. $i \neq 1$) and the induction assumption on n for $\ell_j = p$ (i.e. $i = 1$). By the definition of the measure $\mathbb{Q}_{n_0}^{(k)}$, we have

$$\begin{aligned}
\frac{\mu^{p(n-n_0)} \mathbf{Q}_{n_0}^{(p)}(r_n(\tau) = \mathbf{t})}{\mathbb{P}_{n_0}(r_n(\tau) = t)} &= p! \sum_{i=1}^{k_0 \wedge p} \frac{\phi(a/\mu^{n_0+1})^{k_0-i}}{\phi^{(p)}(a/\mu^{n_0})} \sum_{(\ell_1, \dots, \ell_i) \in S_{i,p}} \left(\prod_{j=1}^i \frac{\phi^{(\ell_j)}(a/\mu^{n_0+1})}{\ell_j!} \right) \\
&\quad \times \sum_{1 \leq r_1 < \dots < r_i \leq k_0} \left(\prod_{j=1}^i \frac{\ell_j!}{\phi^{(\ell_j)}(a/\mu^{n_0+1})} \sum_{s=1}^{\ell_j} a_s^{(\ell_j)}(n) \phi(a/\mu^n)^{z_n(\mathbf{t}_{r_j})-s} H_s(z_n(\mathbf{t}_{r_j})) \right) \\
&\quad \times \left(\prod_{\substack{1 \leq k \leq k_0 \\ k \notin \{r_1, \dots, r_i\}}} \frac{\phi(a/\mu^n)^{z_n(\mathbf{t}_k)}}{\phi(a/\mu^{n_0+1})} \right) \\
&= \frac{p!}{\phi^{(p)}(a/\mu^{n_0})} \sum_{i=1}^{k_0 \wedge p} \sum_{(\ell_1, \dots, \ell_i) \in S_{i,p}} \sum_{1 \leq r_1 < \dots < r_i \leq k_0} \phi(a/\mu^n)^{z_n(\mathbf{t})} \\
&\quad \times \prod_{j=1}^i \sum_{s=1}^{\ell_j} a_s^{(\ell_j)}(n) \phi(a/\mu^n)^{-s} H_s(z_n(\mathbf{t}_{r_j}))
\end{aligned}$$

using that $\sum_{k=1}^{k_0} z_n(\mathbf{t}_k) = z_n(\mathbf{t})$

$$\begin{aligned}
&= \frac{p!}{\phi^{(p)}(a/\mu^{n_0})} \sum_{i=1}^{k_0 \wedge p} \sum_{(\ell_1, \dots, \ell_i) \in S_{i,p}} \sum_{1 \leq r_1 < \dots < r_i \leq k_0} \phi(a/\mu^n)^{z_n(\mathbf{t})} \\
&\quad \times \sum_{w=i}^p \sum_{(s_1, \dots, s_i) \in S_{i,w}^+} \prod_{j=1}^i a_{s_j}^{(\ell_j)}(n) \phi(a/\mu^n)^{-s_j} H_{s_j}(z_n(\mathbf{t}_{r_j})) \\
&= \frac{p!}{\phi^{(p)}(a/\mu^{n_0})} \sum_{w=1}^p \sum_{i=1}^{k_0 \wedge w} \sum_{1 \leq r_1 < \dots < r_i \leq k_0} \sum_{(s_1, \dots, s_i) \in S_{i,w}} \phi(a/\mu^n)^{z_n(\mathbf{t})-w} \\
&\quad \times \left(\sum_{(\ell_1, \dots, \ell_i) \in S_{i,p}} \prod_{j=1}^i a_{s_j}^{(\ell_j)}(n) \right) \prod_{j=1}^i H_{s_j}(z_n(\mathbf{t}_{r_j})).
\end{aligned}$$

Using successively Lemma 4.1 and Lemma 6.1 gives

$$\begin{aligned}
&\frac{\mu^{p(n-n_0)} \mathbf{Q}_{n_0}^{(p)}(r_n(\tau) = \mathbf{t})}{\mathbb{P}_{n_0}(r_n(\tau = t))} \\
&= \frac{p!}{\phi^{(p)}(a/\mu^{n_0})} \sum_{w=1}^p a_w^{(p)}(n) \phi(a/\mu^n)^{z_n(\mathbf{t})-w} \\
&\quad \times \left(\sum_{i=1}^{k_0 \wedge w} \sum_{1 \leq r_1 < \dots < r_i \leq k_0} \sum_{(s_1, \dots, s_i) \in S_{i,w}^+} \prod_{j=1}^i H_{s_j}(z_n(\mathbf{t}_{r_j})) \right) \\
&= \frac{p!}{\phi^{(p)}(a/\mu^{n_0})} \sum_{w=1}^p a_w^{(p)}(n) \phi(a/\mu^n)^{z_n(\mathbf{t})-w} H_w(z_n(\mathbf{t}))
\end{aligned}$$

which ends the induction. $\square$

5. THE SUB-CRITICAL CASE

In this section, we consider a sub-critical offspring distribution q and we assume that there exists $\kappa > 1$ such that $f(\kappa) = \kappa$ and $f'(\kappa) < +\infty$ (this implies in particular that q admits moments of any order).

We define $\bar{f}(t) = f(\kappa t)/\kappa$ for $t \in [0, 1]$ and note that $\bar{f}$ is the generating function of a super-critical offspring distribution $\bar{q}$ with $\bar{q}_n = \kappa^{n-1} q_n$. The mean $\bar{\mu}$ of $\bar{q}$ is $f'(\kappa)$, the smallest positive fixed point of $\bar{f}$ is $\bar{\kappa} = 1/\kappa$ and $\bar{f}'(\bar{\kappa}) = \mu$. Let $\bar{\tau}$ be the corresponding genealogical tree. It is elementary to check that, for every $n \in \mathbb{N}$ and nonnegative measurable function φ , we have

$$(33) \quad \mathbb{E}[\varphi(r_n(\bar{\tau}))] = \mathbb{E}[\kappa^{Z_n-1} \varphi(r_n(\tau))].$$

We deduce from Theorem 3.3, Theorem 4.2 and 4.9 the following result:

Theorem 5.1. *Let $p \in \mathbb{N}$. Let q be a sub-critical offspring distribution with generating function f and suppose that there exists a unique $\kappa > 1$ such that $f(\kappa) = \kappa$ and $f^{(p \vee 1)}(\kappa) < +\infty$. Then for every $n \in \mathbb{N}$ and every $\Lambda_n \in \mathcal{F}_n$, we have*

- For every $s \in [0, \kappa)$,

$$\lim_{m \rightarrow +\infty} \frac{\mathbb{E}[H_p(Z_{m+n})s^{Z_{m+n}}\mathbf{1}_{\Lambda_n}]}{\mathbb{E}[H_p(Z_{m+n})s^{Z_{m+n}}]} = \begin{cases} \mathbb{E}[\mathbf{1}_{\Lambda_n}] & \text{if } p = 0, \\ \mathbb{E}\left[\frac{Z_n}{\mu^n}\mathbf{1}_{\Lambda_n}\right] & \text{if } p \geq 1. \end{cases}$$

- For every $a \geq 0$,

$$\lim_{m \rightarrow +\infty} \frac{\mathbb{E}\left[H_p(Z_{m+n})\kappa^{Z_{m+n}}e^{-a\frac{Z_{m+n}}{f'(\kappa)^{m+n}}}\mathbf{1}_{\Lambda_n}\right]}{\mathbb{E}\left[H_p(Z_{m+n})\kappa^{Z_{m+n}}e^{-a\frac{Z_{m+n}}{f'(\kappa)^{m+n}}}\right]} = \mathbb{E}\left[\frac{1}{f'(\kappa)^{pn}}\kappa^{Z_n-1}\bar{G}_n^{(p)}(Z_n)\right] := \mathbb{E}\left[\bar{M}_n^{(p)}\mathbf{1}_{\Lambda_n}\right]$$

where $\bar{G}_n^{(p)}$ is the function defined by (25) associated with the offspring distribution $\bar{q}$.

Moreover, the probability measure $\mathbb{Q}_{n_0}^{(p)}$ defined by (30) with $M^{(p)}$ replaced by $\bar{M}^{(p)}$, is the probability measure $\bar{\mathbf{Q}}_{n_0}^{(p)}$ defined in Definition 4.4 with q replaced by $\bar{q}$.

Proof. We only prove one case, the other ones can be handled in the same way.

Let us consider $p \geq 1$ and $s \in [0, \kappa)$. Using Equation (33) then Theorem 3.3 (remark that, as q is sub-critical, $q_0 > 0$), and then Equation (33) again, we have

$$\begin{aligned} \lim_{m \rightarrow +\infty} \frac{\mathbb{E}[H_p(Z_{m+n})s^{Z_{m+n}}\mathbf{1}_{\Lambda_n}]}{\mathbb{E}[H_p(Z_{m+n})s^{Z_{m+n}}]} &= \lim_{m \rightarrow +\infty} \frac{\mathbb{E}[H_p(\bar{Z}_{m+n})(s/\kappa)^{\bar{Z}_{m+n}}\mathbf{1}_{\Lambda_n}]}{\mathbb{E}[H_p(\bar{Z}_{m+n})(s/\kappa)^{\bar{Z}_{m+n}}]} \\ &= \mathbb{E}\left[\frac{\bar{Z}_n\bar{\kappa}^{\bar{Z}_n-1}}{\bar{f}'(\bar{\kappa})^n}\mathbf{1}_{\Lambda_n}\right] = \mathbb{E}\left[\frac{\bar{Z}_n}{\kappa^{\bar{Z}_n-1}\mu^n}\mathbf{1}_{\Lambda_n}\right] = \mathbb{E}\left[\frac{Z_n}{\mu^n}\mathbf{1}_{\Lambda_n}\right]. \end{aligned}$$

□

6. APPENDIX: A TECHNICAL LEMMA ON THE HILBERT POLYNOMIALS

Lemma 6.1. *For every $w \geq 1$, for every $k \geq 2$ and every integers $(t_1, \dots, t_i)$, we have*

$$H_w\left(\sum_{j=1}^k t_j\right) = \sum_{i=1}^{w \wedge k} \sum_{1 \leq r_1 < \dots < r_i \leq k} \sum_{(s_1, \dots, s_i) \in S_{i,w}} \prod_{j=1}^i H_{s_j}(t_{r_j})$$

Proof. We prove this formula by induction on k .

First, for $k = 2$, the right-hand side of the equation is, for every $w \geq 2$ (the formula is obvious for $w = 1$),

$$\begin{aligned} \sum_{i=1}^2 \sum_{1 \leq r_1 < \dots < r_i \leq 2} \sum_{(s_1, \dots, s_i) \in S_{i,w}} \prod_{j=1}^i H_{s_j}(t_{r_j}) &= H_w(t_1) + H_w(t_2) + \sum_{s_1=1}^{w-1} H_{s_1}(t_1)H_{w-s_1}(t_2) \\ &= \binom{t_1}{w} + \binom{t_2}{w} + \sum_{s_1=1}^{w-1} \binom{t_1}{s_1} \binom{t_2}{w-s_1} \\ &= \sum_{s_1=0}^w \binom{t_1}{s_1} \binom{t_2}{w-s_1} = \binom{t_1+t_2}{w} = H_w(t_1+t_2). \end{aligned}$$

Assume now that the formula of the lemma is true for every $2 \leq k$ and let us prove it for $k + 1$. We have, using first the formula for $k = 2$,

$$\begin{aligned}
H_w \left(\sum_{j=1}^{k+1} t_j \right) &= H_w \left(\sum_{j=1}^k t_j + t_{k+1} \right) \\
&= H_w \left(\sum_{j=1}^k t_j \right) + H_w(t_{k+1}) + \sum_{s=1}^{w-1} H_s \left(\sum_{j=1}^k t_j \right) H_{w-s}(t_{k+1}) \\
&= \sum_{i=1}^{w \wedge k} \sum_{1 \leq r_1 < \dots < r_i \leq k} \sum_{(s_1, \dots, s_i) \in S_{i,w}} \prod_{j=1}^i H_{s_j}(t_{r_j}) + \\
&\quad H_w(t_{k+1}) + \sum_{s=1}^{w-1} \sum_{i=1}^{s \wedge k} \sum_{1 \leq r_1 < \dots < r_i \leq k} \sum_{(s_1, \dots, s_i) \in S_{i,s}} \left(\prod_{j=1}^i H_{s_j}(t_{r_j}) \right) H_{w-s}(t_{k+1})
\end{aligned}$$

by the induction assumption. Inverting the sums in the last term and setting $s_{i+1} = w - s$ than $i' = i + 1$ yields

$$\begin{aligned}
H_w \left(\sum_{j=1}^{k+1} t_j \right) &= \sum_{i=1}^{w \wedge k} \sum_{1 \leq r_1 < \dots < r_i \leq k} \sum_{(s_1, \dots, s_i) \in S_{i,w}} \prod_{j=1}^i H_{s_j}(t_{r_j}) + \\
&\quad H_w(t_{k+1}) + \sum_{i=1}^{(w-1) \wedge k} \sum_{1 \leq r_1 < \dots < r_i \leq k} \sum_{s=i}^{w-1} \sum_{(s_1, \dots, s_i) \in S_{i,s}} \left(\prod_{j=1}^i H_{s_j}(t_{r_j}) \right) H_{w-s}(t_{k+1}) \\
&= \sum_{i=1}^{w \wedge k} \sum_{1 \leq r_1 < \dots < r_i \leq k} \sum_{(s_1, \dots, s_i) \in S_{i,w}} \prod_{r=1}^i H_{s_j}(t_{r_j}) + \\
&\quad H_w(t_{k+1}) + \sum_{i'=2}^{w \wedge (k+1)} \sum_{1 \leq r_1 < \dots < r_{i'-1} \leq k} \sum_{(s_1, \dots, s_{i'}) \in S_{i',w}} \left(\prod_{j=1}^{i'-1} H_{s_j}(t_{r_j}) \right) H_{s_{i'}}(t_{k+1}) \\
&= \sum_{i=1}^{w \wedge (k+1)} \sum_{1 \leq r_1 < \dots < r_i \leq k+1} \sum_{(s_1, \dots, s_i) \in S_{i,w}} \prod_{j=1}^i H_{s_j}(t_{r_j})
\end{aligned}$$

which is the looked after formula. □

ACKNOWLEDGEMENTS

The authors want to thank Luc Hillairet for several helpful discussions and his indications to obtain Proposition 4.3 or Lemma 4.1.

REFERENCES

- [1] R. Abraham and J.-F. Delmas. Local limits of conditioned Galton-Watson trees: the condensation case. *Electron. J. Probab.*, 19:no. 56, 29, 2014.
- [2] R. Abraham and J.-F. Delmas. Local limits of conditioned Galton-Watson trees: the infinite spine case. *Electron. J. Probab.*, 19:no. 2, 19, 2014.
- [3] R. Abraham and J.-F. Delmas. Asymptotic properties of expanding Galton-Watson trees. arXiv:1712.04650, 2017.

- [4] K. B. Athreya and P. E. Ney. *Branching processes*. Dover Publications, Inc., Mineola, NY, 2004. Reprint of the 1972 original [Springer, New York; MR0373040].
- [5] P. Debs. Penalisation of the standard random walk by a function of the one-sided maximum, of the local time, or of the duration of the excursions. In *Séminaire de Probabilités XLII*, volume 1979 of *Lecture Notes in Math.*, pages 331–363. Springer, Berlin, 2009.
- [6] P. Debs. Penalisation of the symmetric random walk by several functions of the supremum. *Markov Process. Related Fields*, 18(4):651–680, 2012.
- [7] K. Fleischmann and V. Wachtel. On the left tail asymptotics for the limit law of supercritical Galton-Watson processes in the Böttcher case. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(1):201–225, 2009.
- [8] S. C. Harris and M. I. Roberts. The many-to-few lemma and multiple spines. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(1):226–242, 2017.
- [9] S. Janson. Simply generated trees, conditioned Galton-Watson trees, random allocations and condensation. *Probab. Surv.*, 9:103–252, 2012.
- [10] S. Karlin and J. McGregor. Embedding iterates of analytic functions with two fixed points into continuous groups. *Trans. Amer. Math. Soc.*, 132:137–145, 1968.
- [11] H. Kesten. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(4):425–487, 1986.
- [12] H. Kesten, P. E. Ney, and F. L. Spitzer. The galton-watson process with mean one and finite variance. *Teor. Veroyatnost. i Primenen.*, 11(4):579–611, 1966.
- [13] Y.-X. Ren, S. R., and S. S. A 2-spine decomposition of the critical Galton-Watson tree and a probabilistic proof of Yaglom’s theorem. ArXiv 1706.07125, 2017.
- [14] B. Roynette, P. Vallois, and M. Yor. Limiting laws associated with Brownian motion perturbed by its maximum, minimum and local time. II. *Studia Sci. Math. Hungar.*, 43(3):295–360, 2006.
- [15] B. Roynette, P. Vallois, and M. Yor. Some penalisations of the Wiener measure. *Jpn. J. Math.*, 1(1):263–290, 2006.
- [16] B. Roynette, P. Vallois, and M. Yor. Brownian penalisations related to excursion lengths. VII. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(2):421–452, 2009.
- [17] B. Roynette and M. Yor. *Penalising Brownian Paths*. 2009.
- [18] E. Seneta. Functional equations and the Galton-Watson process. *Advances in Appl. Probability*, 1:1–42, 1969.

INSTITUT DENIS POISSON, UNIVERSITÉ D’ORLÉANS, UNIVERSITÉ DE TOURS, CNRS, RUE DE CHARTRES,
B.P. 6759, 45067 ORLÉANS CEDEX 2, FRANCE.

E-mail address: romain.abraham@univ-orleans.fr

E-mail address: pierre.debs@univ-orleans.fr